Semi-global analysis of Lotka-Volterra systems with constant terms

Submitted by
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The miracle is not that we do this work,
but that we are happy to do it.
As we can do no great things,
only small things with great love.
MOTHER TERESA

Contents

Summary

This thesis is concerned with Lotka–Volterra systems with constant terms. We focus on semi-global analysis, which is a tool to qualitatively classify the behaviour of the solutions of a dynamical system.

We are first concerned with the bifurcation analysis of two-dimensional Lotka-Volterra systems with a constant term. We investigate unusual bifurcations that occur in the parameter space. The organizing center of the bifurcation diagram will be a transcritical bifurcation curve, interacting with two saddle-node bifurcation curves. These interactions give us two unusual bifurcations that seem not to have been analysed before.

The previous analysis motivated us to do a bifurcation analysis of systems having the special structure that the two-dimensional Lotka-Volterra systems with constant terms have, i.e. a codimension-one invariant manifold. We identify and analyse all the codimension-one and codimension-two bifurcations in a similar way as bifurcation analysis of a general system is done. In this way, the Lotka-Volterra systems with constant terms are just examples of general systems having a special structure.

Finally we are concerned with the existence of first integrals of Lotka–Volterra systems with constant terms. We mainly discuss two-dimensional and three-dimensional Lotka-Volterra systems. Conditions on the parameters are obtained in order to guarantee that the addition of the constant terms still gives the existence of first integrals of Lotka-Volterra systems.

Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis submitted for the award of any other degree or diploma.

No other person's work has been used without due acknowledgment in the main text of the thesis.

This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

Kie Van Ivanky Saputra
August 2008

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Kie Van Ivanky Saputra

Be joyful always;
pray continually;
give thanks in all circumstances,
for this is God's will for you in Christ Jesus.
1 Thessalonians 5:16-18

List of Tables

List of Figures

CHAPTER 1

Introduction

1.1 History of the Lotka-Volterra model and population dynamics

We start by giving a bit of history of the Lotka-Volterra model which was founded by Vito Volterra (1860-1940) and Alfred J. Lotka (1880-1949)[1]. Vito Volterra was an Italian mathematician who retired from a distinguished career in pure mathematics in the early 1920s. Vito Volterra's son in law, Humberto D'Ancona, who was a biologist, studied populations of various species of fish in the Adriatic Sea. In 1926, he conducted a statistical study of the number of each species sold on fish markets of three ports: Fiume, Trieste, and Venice and noticed that during World War I (as we now call it), the number of predators among Adriatic fauna had increased while the number of prey had diminished. He concluded that this seemed to be a consequence of the reduction of fishing due to the hostilities between Italy and Austria. However, he was wondering why it worked in this way and not in another. Having no biological or ecological explanation for this phenomenon, he asked Volterra if Volterra could come up with a mathematical model that might explain what was going on. After months, Volterra developed a series of models for interactions of two or more species (see Kingsland [69]). From that time on, Volterra devoted his studies to models in ecology. (For a nice treatment of Volterra's works in ecology, see Volterra [124] or a collection of studies by Volterra et al. [111]).

Meanwhile, Alfred J. Lotka (1880-1949), who was an American mathematical biologist (and later actuary) formulated many of the same models as Volterra, independently and at about the same time. He published a book titled *Elements of Physical Biology* (see Lotka [88]). His primary example of a predator-prey system comprised a plant population and an herbivorous animal dependent on that plant for food. It is safe to assume that those two were completely unaware of each other's work. The model they came up with is now known as the Lotka-Volterra model.

We shall let $N(t)$ be the prey population density and $P(t)$ be the predator population density. The usual assumption is made here, namely, that the growth rate of any species is

[1] http://www.math.duke.edu/education/ccp/materials/engin/predprey/pred2.html

proportional to the density of that species present at that time. A further general assumption is that the species live in a homogeneous environment, age structures are not taken into account, the prey has unlimited resources, the prey's only threat is the predator, the predator is a specialist (i.e. the predator's only food supply is the prey) and the predator's growth depends on the prey it catches.

For the prey model, it is assumed that the prey growth, if left alone, is malthusian, i.e. the specific growth rate is constant [34, 89, 98]. It is further assumed that the specific growth rate is diminished by an amount proportional to the predator population. For the predator model, it is then assumed that in the absence of prey, predators will become extinct exponentially but their growth rate is enhanced by an amount proportional to the prey population number. This leads to the following model:

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)(\alpha - \beta P(t)), \qquad \alpha, \beta > 0, \\ \frac{dP(t)}{dt} &= P(t)(-\gamma + \delta N(t)), \qquad \gamma, \delta > 0. \end{aligned} \tag{1.1}$$

This model was the first attempt to mathematically represent a population model that achieved a cyclic balance in a population. This model has been analysed by various text books in dynamical systems, mathematical biology, ecology, differential equations etc. [55,62,98,123]. The solution of the model above is oscillatory as shown in the Figures 1.1 and 1.2. As a consequence, the densities of predator and prey will oscillate periodically, with both the amplitude and frequency of the oscillations, as shown in Figure 1.2, depending on initial conditions and parameters $\alpha, \beta, \gamma, \delta$.

We are now ready for Volterra's explanation of the increase of predatory fish during the war. The densities of prey, $N(t)$ and predator, $P(t)$ oscillated periodically but Volterra showed that the temporal averages of $N(t)$ and $P(t)$ remain constant and are equal to $(\gamma/\delta, \alpha/\beta)$

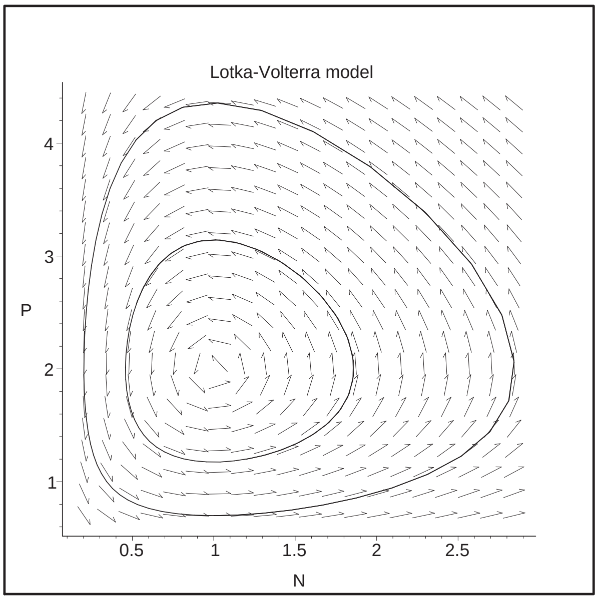

Figure 1.1: The phase portraits of the model (1.1). The data are $\alpha = 2$, $\beta = 1$, $\gamma = 1$ and $\delta = 1$.

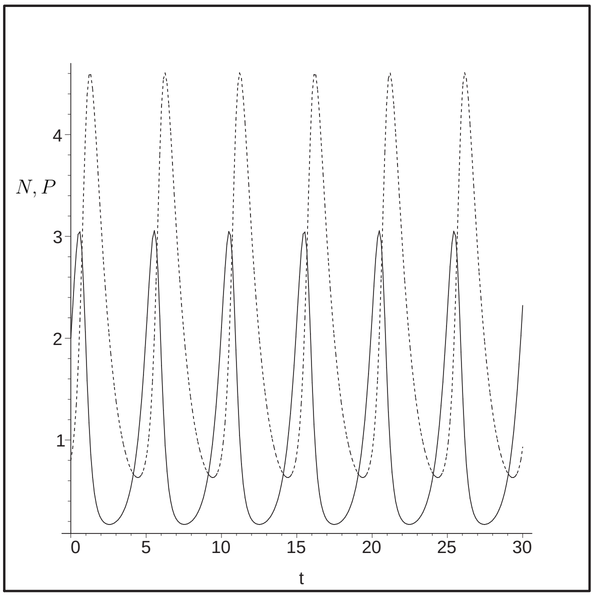

Figure 1.2: Periodic activity of prey (solid line) and predator (dotted line) populations, generated by the Lotka-Volterra model (1.1). The data are the same as those of Figure 1.1.

respectively. The supplementary contribution of fishing diminishes the quantity of α (the rate of increase of the prey in the absence of predators) and increases the quantity of γ (the rate of decrease of predators in the absence of prey). However, fishing does not affect the values of β and δ, which measure the effects of the interaction between predators and their prey. Thus, the time average of the population number of prey is now larger than in the unperturbed case. In contrast, the time average of the population number of predators is now smaller than in the unperturbed case, leading to an increase of predators and a decrease of prey that are just what D'Ancona observed.

The model (1.1) has also been derived independently in the following fields:

1. epidemics (see Kermack and McKendrick [67, 68]), with $\alpha = 0$ and
    - N are susceptible individuals and
    - P are infective individuals,
2. ecology (Lotka [88] and Volterra [124]), with
    - N are prey and
    - P are predators,
3. combustion theory (see Hoppensteadt [61]), with
    - N and P are chemical radicals formed during H_2 and O_2 combustion,
4. economics (see Hoppensteadt [61]), with
    - N are the populace and
    - P are predatory institution,

and numerous studies from diverse disciplines.

The Lotka-Volterra model (1.1), from many points of view is unstable and unrealistic. Firstly, in the absence of predators, the population of prey would grow exponentially towards infinity. This feature is easily corrected, one way is to introduce a competition rate within the prey species. We also assume that the density of prey, in the absence of predators, follows the logistic model,

$$\frac{dN(t)}{dt} = rN(t)(1 - \frac{N(t)}{K}), \tag{1.2}$$

where r and K are positive constants. The constant r is the growth rate of the prey while the constant K is the carrying capacity of the environment that limits the number of prey population that can be considered as the competition rate as well. This logistic model was first proposed by Verhulst in 1838 (see Murray [98]) to adjust the exponential growth of the population model at that time. We then can add another term into the model (1.1) as an intraspecific competition and if we wish, we may also allow an intraspecific competition within the predators. The latter is less crucial, as their population does not explode anyway. This leads to a more general Lotka-Volterra model as follows,

$$\begin{aligned}
\frac{dN(t)}{dt} &= N(t)(\alpha - \eta N(t) - \beta P(t)), \qquad \alpha, \beta, \eta > 0,\\
\frac{dP(t)}{dt} &= P(t)(-\gamma + \delta N(t) - \kappa P(t)), \qquad \gamma, \delta, \kappa > 0.
\end{aligned} \tag{1.3}$$

Thus the classical model (1.1) is one special case of the above model.

Another reason why the classical model is unrealistic is the fact that the solution oscillates periodically in the same periodic solution all the time. If a prey population increases, it encourages growth of its predator. More predators however consume more prey, the population of which starts to decline. With less food around, the predator population declines and when it is low enough, this allows prey population to increase and the whole cycle starts over again. Depending on the detailed system, such oscillations can grow or decay or go into a stable limit cycle oscillation, which does not occur in either (1.1) or (1.3).

Georgii Frantsevitch Gause proposed another system of much more general equations [41, 42], which using modern notations x and y, take the following form:

$$\begin{aligned}
\dot{x} &= xg(x) - yp(x),\\
\dot{y} &= y(-\gamma + q(x)),
\end{aligned} \tag{1.4}$$

where $\dot{x}$, $\dot{y}$ represent first derivatives of x and y with respect to time. Here $g(x)$ is the specific growth rate of the prey in the absence of any predators and $p(x), q(x)$ are the response functions for the predator with respect to that particular prey. The former function is positive on an interval $[0, K]$ and negative for $x > K$ (because, for example, the food resources are limited, or there is an intraspecific competition within the prey). We suppose that p is a positive function with $p(0) = 0$, while q is strictly increasing for $x > 0$, has a negative limit when x decreases to 0 and a positive limit when x increases to $+\infty$. These models are more reasonable and more flexible than (1.1) and (1.3). In Gause's model, he assumed that $q(x) = cp(x)$ for some constant

c since essentially, $q(x)$ will have properties similar to $p(x)$. We refer to Freedman [34] and Sigmund [112] for more explanation of the Gause's model. Note that $\gamma = 0$ in Sigmund [112].

Another improvement of predator and prey population models was done by Kolmogorov. After noticing population models in Volterra's work, Kolmogorov considered the most general case possible [34],

$$\begin{aligned} \dot{x} &= xS(x,y), \\ \dot{y} &= yW(x,y). \end{aligned} \tag{1.5}$$

Conditions must be put on S and W to make x and y a prey and a predator respectively [112]. The first group of conditions requires that, if the number of predators increases, then the rates of increase of the two populations decrease;

$$\frac{\partial S}{\partial y} < 0 \quad \text{and} \quad \frac{\partial W}{\partial y} < 0.$$

In addition, the rate of increase of the predator population increases with respect to the increase of population of prey, while the rate of the prey population decreases,

$$\frac{\partial S}{\partial x} < 0 \quad \text{and} \quad \frac{\partial W}{\partial x} > 0.$$

Of course, there are more conditions we need in order to approximate the model closely to the reality.

The Lotka-Volterra predator-prey model and the other population models are special cases of models in population dynamics. This is because in ecology, interactions between species can be very complex, even when only two species are considered. Each species can affect other species' environment[2]; positively (+), negatively (–), or have no effect (0). Major categories include:

- mutualism (++),
- commensalism (+0),
- predator/prey (+–),
- competition (−−), and
- amensalism (rare) (–0).

As in the predator-prey models, the simplest model that can represent all the above two-species interactions is of the Lotka-Volterra type. Thus, using a modern notation we have the general two-dimensional Lotka-Volterra model that can represent all interactions between two species:

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2), \\ \dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2), \end{aligned} \tag{1.6}$$

[2]http://ipmworld.umn.edu/chapters/ecology.htm

where x_1 and x_2 denote the two species. The signs of b_i and a_{ij} $(i, j = 1, 2)$ determine what population model we have. For example, we can construct a competition model here by letting b_1, b_2 be positive and a_{11}, a_{12}, a_{21} and a_{22} be negative. Among all two-species interactions, predator-prey and competition relationships are the most studied models in population dynamics. Besides the Lotka-Volterra model, the Kolmogorov model can also be used to represent not only predator-prey interaction but also competition and cooperative relationships using the freedom in the conditions of functions S and W (see equation (1.5)). In the Gause model, one has to generalise some assumptions for this model to be able to model not only predator-prey relationship but all two-species interactions as well.

There has been a lot of mathematical analysis performed to analyse the general two-dimensional Lotka-Volterra model (1.6) and the other population models such as Gause (1.4) and Kolmogorov (1.5). We first start to discuss various mathematical analyses that have been performed to the Lotka-Volterra type model. One of the earliest mathematical methods to study the Lotka-Volterra system is using replicator dynamics. The replicator equation arises in the game theoretical model for the evolution of behaviour in animal conflicts with dynamics (Hofbauer et al. [57]). It is shown in Hofbauer [56] that the replicator equation for $(n+1)$ strategies corresponds to the generalized Lotka-Volterra equation for n-populations,

$$\begin{aligned}
\dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n),\\
\dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n),\\
&\vdots\\
\dot{x}_n &= x_n(b_n + a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n).
\end{aligned} \tag{1.7}$$

It is also shown in Bomze [10] that using a replicator equation for three strategies, one can obtain a two-dimensional classification of Lotka-Volterra models (1.6). The connection of Lotka-Volterra models and replicator equations is also discussed in the unpublished thesis [90]. A more detailed introduction of this connection has been summarized in a book by Hofbauer and Sigmund [58].

Besides the classification that has been done by means of replicator equations, the Lotka-Volterra equation has also been analysed to see whether it has a limit cycle. There have been a lot of investigations regarding this issue. Using the Bendixson-Dulac theorem, (see Wiggins [126]), it has been shown that the two-dimensional Lotka-Volterra model (1.6) cannot admit a limit cycle (see the work by Coppel [26] and Reyn [26]). However, if we extend this model to a higher dimensional model, it may have a limit cycle. Hofbauer and So [59] considered three-dimensional Lotka-Volterra systems with an equilibrium in the positive octant. The authors showed that these systems have more than one limit cycle around the equilibrium point. The authors, at the end of this paper, conjectured that two is the maximum number of limit cycles in these systems.

After investigating the occurrence of limit cycles in the Lotka-Volterra system, people are also curious whether the Lotka-Volterra system admits a first integral (i.e. a function that is constant along the solution of the vector field). The search for first integrals is one classical tool in the classification of the solution of dynamical systems, in particular Lotka-Volterra models.

One of the earliest attempt to find first integrals was the Carleman embedding method that was used by Cairó et al. [15, 18]. Instead of finding the first integral, the authors obtained a more general function that is called an invariant. This function depends on time not like a first integral, which does not depend on time.

The Darboux method has also been used to find a first integral of the two-dimensional Lotka-Volterra system (1.6) [20]. This method links the theory of algebraic solutions of differential equations to the search of the first integral or the integrating factor. Extensive results have been found by Cairó and Llibre [22] in which algebraic solutions of degree one to four have been found. Consequently, the first integral or the integrating factor can be found. The integrating factor, in the end, can be used to find the first integral even when it is not trivial. Polynomial inverse integrating factors have been introduced by Cairó et al. [24] to show the integrability of Lotka-Volterra models via polynomial first integrals. Finally a more recent result in the integrability of Lotka-Volterra systems is found by Cairó et al. [21] where the complete classification of Liouvillian first integrals for the quadratic Lotka-Volterra model was presented. More detail about the Liouvillian first integral can be found in a paper by Singer [113]. Another recent result is found in Llibre and Valls (2007) [87], where the authors provided a complete classification of all Lotka-Volterra systems having a global first integral.

All results listed above are discussing the two-dimensional Lotka-Volterra model except the ones that were using the Carleman embedding method, in which the authors studied n-dimensional models. In the three-dimensional model, there have been a lot of discussions as well, see Grammaticos [46], Labrunie [81] and Moulin Ollagnier [96]. The authors have studied the integrability of three-dimensional Lotka-Volterra models that depend on three parameters via polynomial first integrals. The model that is discussed is a special case of general Lotka-Volterra system (1.7) for $n = 3$:

$$\begin{aligned} \dot{x} &= x(Cy + z), \\ \dot{y} &= y(Az + x), \\ \dot{z} &= z(Bx + y). \end{aligned} \tag{1.8}$$

Darboux integrability has also been used to show the integrability of three-dimensional systems, Cairó and Llibre [14, 23]. Also, Moulin Ollagnier [97] classified conditions for the three parameters for which the three-dimensional Lotka-Volterra systems have a Liouvillian first integral of degree zero.

Other methods such as Hamiltonian method [16,38,63,64], direct and indirect integrating method [37, 39] have also been applied to find integrals of two and three-dimensional Lotka-Volterra systems.

Other than first integrals, other geometrical structures in Lotka-Volterra models have also been investigated. It has been shown in Schimming [110] that necessary and sufficient conditions can be found on the Lotka-Volterra models to admit a conservation law. The definition of *conservative* can be found in that paper. Hamiltonian structure is also investigated by Plank [102]. The author derived conditions for two-dimensional Lotka-Volterra equations to have a Hamiltonian structure in the first part of his paper. The second part discussed that the Hamiltonian structure that the author derived in the two-dimensional case can also be used as an Ansatz for possible Hamiltonian functions and invariants of the n-dimensional case. Another result concerning the Hamiltonian structure is also discussed by the same author, Plank [103], in which the author discussed the dynamics of n-dimensional Lotka-Volterra system having an invariant hyperplane.

Finally, in McLachlan and Quispel [93, sec 3.14] and references therein, various special structures of Lotka-Volterra systems are discussed. Instead of writing the n-dimensional model as in (1.7), the authors wrote the system (1.7) as follows:

$$\dot{x}_i = x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j \right), \qquad i = 1, .., n. \tag{1.9}$$

In the domain $x_i > 0$, the authors defined $u_i := \log x_i$, to get:

$$\dot{u}_i = b_i + \sum_{j=1}^{n} a_{ij} \mathrm{e}^{u_j} \tag{1.10}$$

or

$$\dot{\mathbf{u}} = \mathbf{b} + A\mathrm{e}^{\mathbf{u}}, \tag{1.11}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$ and $A = [a_{ij}]$. Each Lotka-Volterra system falls into one or both of the following cases:

1. $\mathbf{b} \in \mathrm{range}(A)$,

2. $\mathrm{rank}(A) < n$.

In case (1), the system (1.11) can be re-written in a linear gradient form [94]:

$$\dot{\mathbf{u}} = A\nabla V(\mathbf{u}), \tag{1.12}$$

with $V(\mathbf{u}) = \sum_i (\mathrm{e}^{u_i} + c_i u_i)$. Some special cases are:

1. if A is symmetric positive definite, (1.12) is a gradient system;

2. if $A + A^T$ is negative definite ([111, 124]), then (1.12) has V as a Lyapunov function;

3. if A is antisymmetric, (1.12) is either Hamiltonian system (if $\mathrm{rank}(A) = n$) or a Poisson system (if $\mathrm{rank}(A) < n$)

4. if $a_{ii} = 0$ for all i, then (1.12) is divergence free [111, 124].

In recent investigations, it is shown that many economic, physical, and biological phenomena are best represented via difference equations instead of differential equations, Agarwal [2]. There are also situations for which differential equations are the best fit, however the solution of the differential equations is hard to get. One may then use some numerical scheme to transform the given differential equation into a difference equation, Mickens [95]. The resulting difference equation should be dynamically consistent with its continuous version. Analysis of discrete dynamical systems has been done (see, for instance Wiggins [126]). In the Lotka-Volterra model, there have been a large number of discrete analogs of the continuous Lotka-Volterra model. We refer to textbooks in mathematical biology for further references [32, 33, 92, 98]. A nonstandard discretization has been developed to propose a discrete analog of competitive and cooperative models of Lotka-Volterra type by Liu and Elyadi [85]. It was shown in that paper that the resulting difference equation possesses dynamics that is consistent with the continuous Lotka-Volterra model.

Recently, the discrete version of general Lotka-Volterra model was also investigated by Lie and Xiao [86]. The authors studied:

$$\begin{aligned} x_{n+1} &= x_n + r x_n(1 - x_n) - b x_n y_n \\ y_{n+1} &= y_n + (-d + b x_n y_n). \end{aligned} \tag{1.13}$$

Bifurcation theory is applied in order to show that the discrete version of (1.6) can undergo a series of interesting bifurcations for different values of the parameters. Furthermore, Blackmore et al. [8] found the chaotic behaviour of the discrete version. The authors study iterates of the map which is the right-hand side of the Lotka-Volterra population model. It is shown that the map reduces to logistic maps for certain parameter ranges and so has chaotic behaviour there. The main result consists of the determination of parameter ranges for which the map is an orientation reversing horseshoe map on an invariant set.

Apart from the discrete model, there is also a modification of continuous Lotka-Volterra models. The effect of dispersions of the population species and time delays are both taken into consideration. The result is that the Lotka-Volterra model is governed by a system of reaction-diffusion equations with time delays which is a partial differential equation. For an introduction of these delay and diffusion differential equations, we refer to Gopalsamy [45] and Kuang [75]. He and Gopalsamy [48] have studied a general two-dimensional Lotka-Volterra system with time delays as follows,

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(b_1 + a_{11}x(t-\tau) + a_{12}y(t-\tau)) \\ \frac{dy(t)}{dt} &= y(t)(b_2 + a_{21}x(t-\tau) + a_{22}y(t-\tau)). \end{aligned} \tag{1.14}$$

The authors obtained sufficient conditions for the global attractivity of the positive equilibrium of the delay system. This means that when parameters satisfy such conditions, we have a coexistence of two species. Another Lotka-Volterra model with time delay is also found in the paper by Gopalsamy [44]. The author obtained sufficient conditions for n-dimensional Lotka-Volterra models with time delay such that the solution is oscillatory. Further research on this has been done by Kuang and Smith [77], in which the authors presented sufficient conditions for the global stability of the equilibrium of a non-autonomous Lotka-Volterra system with an infinite delay.

Besides the time delay, people also considered diffusion terms in the early 1980s, when two-species models with diffusion terms have been extensively investigated. Blat and Brown [9] discussed predator-prey and competition models which are diffusion partial differential equations. The authors obtained results on the existence of positive solutions with coexistence of the species. A stable coexistence states in the Lotka-Volterra competition model with diffusions is also investigated by Cosner and Lazer [28] and Korman and Leung [73]. Recent research about three-dimensional Lotka-Volterra models with diffusion and time delay shows that a global asymptotic stability implies the nonexistence of positive steady-state solutions (see Wang [125]).

In all modifications and developments of the Lotka-Volterra model described above, we deal with deterministic models. However, from a stochastic point of view, the Lotka-Volterra model can also be investigated. A more general theory of this research perhaps is a stochastic differential equation theory (see a book by Gard [40]). For our first stochastic Lotka-Volterra model, we refer to Froda and Colavita [35], in which the authors introduced a simple stochastic model that describes the interaction between predators and prey populations. The authors considered the classical Lotka-Volterra model (1.1), and assumed that the deterministic functions $N(t)$ and $P(t)$ are perturbed with random errors as follows,

$$\log X_t = \log N(t) + \epsilon_x, \qquad \log Y_t = \log P(t) + \epsilon_y, \tag{1.15}$$

where ϵ_x and ϵ_y are random variables and symmetrically distributed around zero. Their analyses are based on the fact that the model (1.1) admits closed trajectories. A continuation of this research was done by Froda and Nkurunziza [36]. In this paper, the authors generalised what had been done in [35]. The analyses still considered the oscillatory solution of the deterministic model.

Another approach along this direction is also investigated by Mao et al. [91]. In this paper, the authors discussed the Lotka-Volterra model with each parameter subject to stochastic perturbations. Conditions are established such that solutions stay in the positive quadrant for all future time. By weakening some of their hypotheses, their results are slightly improved by Du and Sam [30] about the Lotka-Volterra system perturbed by a multiplicative random noise. The latter authors also proved that the total population always visits any neighbourhood of the origin.

Extinction of populations that barely happens in the deterministic model is discussed by Klebaner and Lipser [72]. In probabilistic models, the authors questioned whether extinction can occur. It is also shown that this perturbed system converges to a Lotka-Volterra model without noise, and satisfies a large deviation principle. Klebaner et al. [71], made a more detailed study of the stochastic fluctuations of the discrete model around its deterministic limit. In particular, the authors proved a functional central limit theorem. Recent research about the long time behaviour of both deterministic and stochastic Lotka-Volterra systems is given by Rudnicki and Pichór [108]. The authors indicated the differences between the deterministic and stochastic models by studying long-time behaviour of both trajectories and distributions of the solutions.

Besides the Lotka-Volterra model, the predator prey model (1.4) introduced by Gause [41, 42], has also been investigated extensively in the literature. Note that this model generalised the Lotka-Volterra predator-prey model in terms of the choice of the response function. The response function $p(x)$ in Lotka-Volterra models is linear and unbounded, whereas a more reasonable response function should be non-linear and bounded. For more conditions that response functions should behave, we refer to a book by Freedman [34]. Holling [60] introduced a more reasonable response function to model the predator-prey relationship. Since then, various response functions have been introduced and analysed to model various interactions of predator-prey models. For instance, we refer to a paper by Kuang [74], in which the author has studied a response function with strictly concave down isocline and shown the existence of at least three limit cycles in the corresponding model.

Ruan and Xiao [107] studied a global analysis of a predator-prey model of Gause type using a bifurcation theory. The authors used the following response function,

$$p(x) = \frac{mx}{a + bx + cx^2}, \tag{1.16}$$

that is usually called a Holling type-IV function. The authors have studied the case where $b = 0$ and shown that their model exhibited a number of bifurcations. Moreover, it has been shown that a limit cycle cannot coexist with a homoclinic bifurcation for all parameters as a part of a codimension-two bifurcation. This research is extended by Zhu et al. [131] with a positive b. Rothe and Shafer [106] have actually studied the same model with a negative b. Both these papers have performed bifurcation analysis to classify the predator-prey model with Holling type-IV response functions. The authors have shown that if the population of prey is large enough, then the extinction of predators occurs regardless of the initial size of predator population. The authors have also shown that the coexistence of predator and prey can be in the form of a steady-state solution or a periodic solution. More focused research about limit cycles or periodic solutions of the predator-prey model with Holling type-IV response functions has been performed by Xiao and Zhu [129].

However, recent investigations and empirical evidence showed that the most natural system (i.e. approximating the reality) should not be using a response function that is prey-dependent (i.e. $p(x)$). Instead, the response function should depend on a ratio of prey and predator populations (i.e. $p(x/y)$). For more biological explanations, we refer to Akçakaya et al. [3] and extensive references cited therein. Research on the ratio-dependent model has revealed rich interesting dynamics such as deterministic extinctions, existences of multiple attractors, and existences of stable limit cycles. Moreover, it was shown in the paper by Jost et al. [65] that the ratio-dependent model has such a complex dynamics near the origin $(0, 0)$. The authors have studied the analytical behaviour at $(0, 0)$ and demonstrated that this equilibrium can be either a saddle point or an attractor which has an important implication concerning the global behaviour of the model. If the origin is an atractor then we have the case of deterministic extinction.

Kuang and Beretta [76] have studied a ratio-dependent predator-prey model that used the following response function,

$$p(x) = \frac{mx}{a + x}. \tag{1.17}$$

This function is usually called a Holling type-II function (see all the references therein). The authors modified the function into $p(x/y)$ and performed a global qualitative analysis and showed that the positive steady-state solution is asymptotically stable for some parameters.

The authors also gave conditions for the other three equilibria to be globally asymptotically stable. A continuation along this line of research has been performed in the paper by Tang and Zhang [117], in which the authors obtained some conditions on parameters such that the system has a heteroclinic loop.

To conclude the discussion of Gause-type predator-prey population models, we refer to a paper by Cosner et al. [27] for the derivation of various forms of functional responses in predator-prey models. In some conditions where the predators are assumed to have a homogeneous spatial distribution, the suitable functional response is prey-dependent, however if the predators are assumed to form a dense colony in a single (possibly moving) location, or if the region where predators can encounter prey is assumed to be of limited size then the functional response depends on the ratio of prey and predators (i.e. the ratio dependence model).

One more reason why Lotka-Volterra equations have attracted ample attention is because chaotic behaviour may occur in higher-dimensional models. One of the first investigations, showing that the Lotka-Volterra model can exhibit such a behaviour is Smale [114]. The author studied a general competition model and argued that under some conditions, any asymptotic dynamical behaviour is possible for populations of five or more species. The occurrence of chaos through quasi-periodic orbits perhaps was first found by Arneodo et al. [4]. The authors of this paper studied one-parameter families of one class of three-dimensional Lotka-Volterra systems. Hirsch in his series of papers [49–54] has studied a differential equation that is competitive or cooperative and shown that in the Lotka-Volterra competition (or cooperative) equation with $n \leq 3$ (i.e. three or less species interaction), no chaotic behaviours are possible, and thus, the model with $n = 4$ is the simplest example where chaotic solutions are possible, as it was shown by Vano et al. [122].

From the point of view of applications, Lokta-Volterra systems have also intrigued a large number of people. The first application that can be represented by this model is of course a population dynamics model. Moreover, the Lotka-Volterra equation can be used to model a complex interaction between n-species such as food chains. In Figure 1.3, we illustrate complex interactions of three (a, b, c) and four (d, e) species, which can be modelled by the n-dimensional Lotka-Volterra system. As an example, we shall formulate a model representing Figure 1.3(d). We shall call the prey, predator, super predator and great super predator as species 1, 2, 3, and 4 respectively. We then have the following system of differential equations:

$$\begin{aligned}
\dot{x}_1 &= x_1(b_1 - a_{11}x_1 - a_{12}x_2), \\
\dot{x}_2 &= x_2(-b_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3), \\
\dot{x}_3 &= x_3(-b_3 + a_{32}x_1 - a_{33}x_2 - a_{34}x_3), \\
\dot{x}_4 &= x_4(-b_4 + a_{43}x_1 - a_{44}x_2),
\end{aligned} \tag{1.18}$$

where all the coefficients are positive. The constants b_i, a_{ii} are the growth rate of each species and the intraspecific competition within each species respectively. While, the constants a_{ij} for $i \neq j$ are the effect of the species j to the species i. We refer to Hofbauer and Sigmund [58] for more information along this direction.

Besides population dynamics, there are also epidemic problems, combustion theory and economy problems that can be modelled by the Lotka-Volterra equation [61]. Recent research

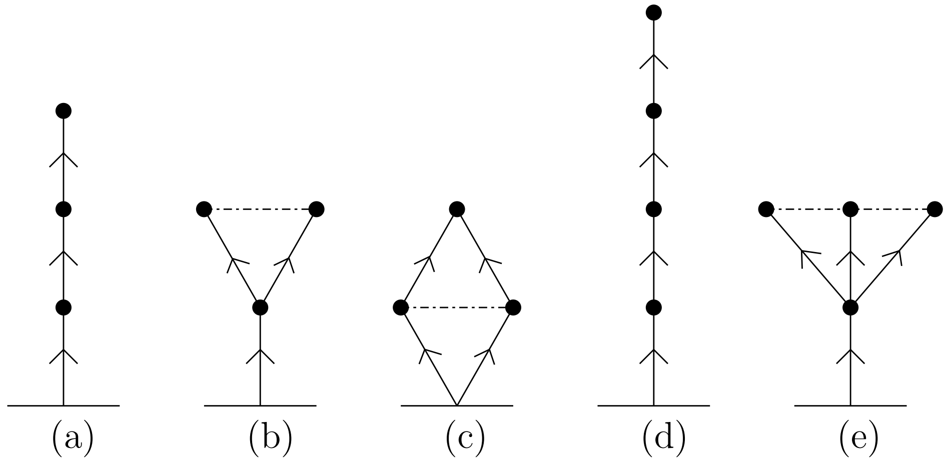

Figure 1.3: Some graphs involving three (a, b, c) and four (d, e) interacting species. A prey is threatened by a predator which is threatened by a super-predator (Figure (a)). Moreover, the super predator is threatened by a great super predator in Figure (d). In Figures (b), (c) and (e), there are dotted horizontal lines representing different species that are on the same level (i.e. they eat the same source). These dotted horizontal lines can be either competition, cooperative or no effect at all.

about neural networks also shows that they can be represented as Lotka-Volterra model with time delays by Yi and Tan [130] and all references therein. Finally, to conclude the discussion of the application of Lotka-Volterra models we refer to the book by Peschel and Mende [101]. The authors of this book attempted to formulate a general theory of growth phenomena from which the authors are able to obtain Lotka-Volterra growth interaction with the help of a large class of equivalence transformations. The authors also covered a wide range of topics, including graph theory, the theory of finite automata, coding theory, vector optimization problems, etc.

In nonlinear population models, a constant-rate is sometimes introduced to represent some of more complex cases in ecology. In populations of one species, a constant term is sometimes added to the population model to represent more dynamics in the model. If the species lives in a specific area and it almost dies out, the positive constant rate serves as migration constants or stocking constants that are done by a human to prevent an extinction. If the species is an animal that is constantly taken by humans for food, then the constant rate can serve as a harvesting effect. This harvesting problem is interesting on its own since there is a question such as what the optimal harvesting rate is, such that we can exclude the probability of extinction. This question is related to a maximum sustainable yield (MSY) according to Clark [25]. If a species is harvested by some process of over exploitation, then this species can become extinct.

For population models of two species or more, the situation is more involved. For instance, in predator-prey type models, if the prey is a dangerous pest, we can increase the predator by adding a constant term to the right hand side of the differential equation describing the predator as a stocking effect in order to control a pest. Or, if the predator is a rare species, we can increase the prey to the system by adding a positive constant term in the prey equation in order to increase the population of predators.

This research perhaps is first done by Brauer and Sánchez [11]. The authors discussed how the stability of an equilibrium is affected by the introduction of the constant term. Various population models that were discussed include: logistic models, modified logistic models and logistic models with time delays and two-dimensional Lotka-Volterra competition models. More advanced techniques are performed in the paper by Xiao and Ruan [128] and Xiao and Jennings [127], in which the authors performed a bifurcation analysis on a Gause-type predator-prey model and found numerous interesting bifurcations. This bifurcation then can be used to find the optimal harvesting as the constant term is one of the parameters that are varied.

Although there has been much progress since the introduction of Lotka-Volterra equations and other population models almost a century ago, certain questions remain unanswered, in particular: to the author's best knowledge, the consequences of constant terms in Lotka-Volterra equations. This thesis will address, amongst other things, the global analysis of general two-dimensional Lotka-Volterra systems with constant terms which we will analyse using a dynamical systems theory.

1.2 Motivation of this thesis

The main topic of this thesis is about the general Lotka-Volterra system with constant terms. As is mentioned in the title of this thesis, we will perform a semi-global analysis of this system. Thus, this thesis is a collection of studies on Lotka-Volterra systems with constant terms. Dynamical systems theory will underline our analysis. We use the word "semi" mainly because dynamical systems theories are very broad and we only use some of those theories to analyse the Lotka-Volterra system with constant terms. Special attention will be given to the following areas in dynamical systems theory in this thesis:

1. bifurcation theory of differential equations,
2. integrability of differential equations.

As is mentioned in the previous section, the Lotka-Volterra system has arisen frequently in mathematical publications. The first issue that we want to discuss concerns two-dimensional Lotka-Volterra systems with a constant term. The discussion involves the theory of bifurcation analysis. Although some aspect of the two-dimensional Lotka-Volterra system with constant terms have been discussed, such as the stability of the positive equilibrium, the stability of the origin (as it represents extinction in the real world), Brauer and Sánchez [11], we believe that a global analysis of general two-dimensional Lotka-Volterra systems with constant terms has not been studied.

Bifurcations of general Lotka-Volterra equations have received less attention in the literature. One of the reasons for this is the structure that is possessed by such systems. In bifurcation theory, the presence of special structure complicates the analysis, since it usually leads to certain degeneracies that one does not normally have. This leads us to then analyse a general dynamical system with the same special structure as the Lotka-Volterra system.

The next issue that is dealt with in this thesis is the existence of first integrals of a dynamical system. It is known that the classical Lotka-Volterra system (1.1) has a first integral for every value of the parameters, in fact it was Volterra himself who proved that it admits a function that is constant with respect to time, (see the book by Hofbauer and Sigmund (1998) [58]). It is also known that the problem of searching for a first integral of general Lotka-Volterra systems (1.7) is still a topic of ongoing research. If we add constant terms to the system, a natural question would be whether the first integral is still preserved and the conditions on the parameters can still be obtained in the presence of the constant terms.

Our goal in this thesis is to provide some mathematical insight into a class of mathematical models of Lotka-Volterra type that arise quite often in the literature. However, we also try to make the connection to other fields as clear as possible. We hope that this work is as enjoyable to read as it was to produce.

1.3 Mathematical preliminaries

In this section, we will describe some mathematical terminology that we are going to use throughout the thesis. We will discuss some concepts that we use in this thesis to understand the Lotka-Volterra system with constant terms and discuss why these aspects are important.

1.3.1 Continuous dynamical systems

In this thesis, we will study differential equations of the following form,

$$\dot{x} = F(x, \mu), \tag{1.19}$$

where $x \in U \subset \mathbb{R}^n$ and $\mu \in V \subset \mathbb{R}^p$, with U and V being open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. The function on the right hand side will always be assumed to be in $\mathcal{C}^r$ (i.e. the derivatives $F', F'', \ldots, F^{(r)}$ exist and are continuous) with r being defined as we go along. The overdot means "$\frac{d}{dt}$". We refer the differential equation (1.19) as a *continuous dynamical system* or a vector field. By a solution of (1.19) we mean a continuously differentiable function, x from some interval $I \subset \mathbb{R}^1$ into $\mathbb{R}^n$, which we represent as follows

$$\begin{aligned} x: \; I &\rightarrow \mathbb{R}^n, \\ t &\mapsto x(t), \end{aligned} \tag{1.20}$$

such that $x(t)$ satisfies (1.19), i.e. $\dot{x}(t) = F(x(t), t, \mu)$. A solution $x(t)$ usually depends on an initial condition $x_0 \in U$, therefore it is common to write the solution of the differential equation as $x(t, x_0)$. We view the variables μ as parameters that are kept constant when we consider the solution of this differential equation. The *phase space* is referred as the space of the dependent variable x. Abstractly, our goal is to understand the geometry of solutions curves in this phase space.

One trivial solution in the continuous dynamical system is called an *equilibrium* which is a point in phase space, x_0 which is kept invariant under the flow of the dynamical system i.e.

$F(x_0) = 0$. Other terms, often used for the term "equilibrium", are "fixed point", "critical point" or "steady state". Another interesting solution is a *periodic solution.* i.e. a non steady-state solution of (1.19) that satisfies $x(t+T) = x(t)$, for a $T \neq 0$.

Let x_0 be an equilibrium. A solution $x(t)$ satisfying

$$\lim_{t\to\pm\infty} x(t) = x_0, \tag{1.21}$$

is called a *homoclinic* solution (homoclinic loop). Another interesting object that we will also see in this thesis is a *heteroclinic* solution (heteroclinic connection). Let x_0 and x_1 be two distinct equilibria and a heteroclinic solution is defined as a non-constant solution, $x(t)$, satisfying

$$\lim_{t\to\infty} x(t) = x_0, \quad \text{and} \quad \lim_{t\to-\infty} x(t) = x_1. \tag{1.22}$$

Remark that homoclinic and heteroclinic solutions only occurs on specific values of parameters. We also want to look for an invariant manifold, i.e. a manifold, $M \subset \mathbb{R}^n$ such that $x(t) \in M$ for all t. The invariant manifold might have a special geometry such as an invariant sphere or an invariant torus.

1.3.2 First integrals

For a general autonomous dynamical system,

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{1.23}$$

where $f(x) = F(x, \mu_0)$, (we fixed $\mu = \mu_0$), a scalar valued function $H(x)$ is said to be a *first integral* if it is constant along the solution $x(t)$ of the differential equation above, i.e.,

$$\frac{dH(x(t))}{dt} = \nabla H(x) \cdot \dot{x} = \nabla H(x) \cdot f(x) = 0,$$

where "$\cdot$" denotes the usual Euclidean inner product. From this relation, we see that the level sets of $H(x)$ (which are generally $(n-1)$-dimensional) are invariant sets. For two-dimensional dynamical systems, the level sets actually give the trajectory of the solution of the system. For this reason, in this thesis we discuss the first integral of the Lotka-Volterra system with constant terms which is in fact two-dimensional. By searching a first integral of this system for some values of the parameters, we can compute the solution of the system.

1.3.3 Stability of an equilibrium

In this section, we would like to briefly discuss the idea of *stability*. The stability of each invariant structure we have described above (i.e. equilibrium, periodic, homoclinic, heteroclinic solutions etc.) is important because it determines the dynamics of solutions of the vector field. Here we use mainly two different stability types: *neutrally stable* (or Lyapunov-stable) and *asymptotically stable*. In a neutrally stable situation, nearby solutions stay close to the invariant structure as time increases while in an asymptotically stable situation, nearby solutions get attracted. In addition, we also have the notion of *local stability* and *global stability*. In this discussion, we restrict ourselves to discuss the local stability of an equilibrium as it will lead to the stability of other invariant structures.

Our first step is to understand the nature of solutions near an equilibrium x_0 with initial conditions close to x_0 (local stability), we will linearize our vector field (1.23). Let,

$$x = x_0 + y. \tag{1.24}$$

Substituting (1.24) into (1.23) and Taylor expanding about x_0 gives,

$$\dot{x} = \dot{x_0} + \dot{y} = f(x_0) + Df(x_0)y + \mathcal{O}(\|y\|^2), \tag{1.25}$$

where Df denotes the first derivative of f and $\|\cdot\|$ is a norm in $\mathbb{R}^n$ (note: in order to obtain (1.25), f must be at least twice differentiable). Using the fact that $\dot{x_0} = f(x_0) = 0$, we have

$$\dot{y} = Df(x_0)y + \mathcal{O}(\|y\|^2). \tag{1.26}$$

The equation above describes the evolution of solutions near x_0, so it seems reasonable to understand the behaviour of the solution close to x_0 by studying the associated linear system,

$$\dot{y} = Df(x_0)y. \tag{1.27}$$

Therefore, the question of stability of x_0 involves the following steps:

1. determine the stability of $y = 0$,
2. show that the stability of $y = 0$ implies the stability of x_0, and
3. determine the stability of $y = 0$ as parameters are varied.

The first step is really an elementary linear differential equation problem, thus if all eigenvalues of $Df(x_0)$ have non-zero real parts, then the fixed point $y = 0$ is hyperbolic, moreover if all eigenvalues of $Df(x_0)$ have negative (positive) real parts then the fixed point $y = 0$ is asymptotically stable (unstable resp.) To answer the second part, we refer to a theorem by Hartman-Grobman (see Guckenheimer and Holmes [47]),

Theorem 1.1 (Hartman-Grobman). *If $Df(x_0)$ has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighbourhood U of x_0 in $\mathbb{R}^n$ locally taking orbits of the vector field (1.23) to those of the linear vector field (1.27). The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

This theorem really means that the stability of the hyperbolic equilibrium solution, x_0 and the nature of solutions near this point are determined by the linearization. However, if the matrix $Df(x_0)$ has an eigenvalue that has a zero real part (non-hyperbolic), then the stability cannot be determined by the linearization. We will discuss the issue regarding the stability of vector fields having a zero real part eigenvalue later on.

The third step is really the main question in bifurcation theory, as when we vary parameters in our dynamical system, the equilibrium might undergo a change of stability and moreover, the qualitative structure of solutions of the vector field might change as well. This phenomenon is known as *bifurcation.* To understand when and how the qualitative structure of solutions of the vector field is changed we define the notion of *conjugacy* and *equivalence* of vector fields. Two vector fields on $\mathbb{R}^n$ are said to be *topologically equivalent* if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that maps orbits of the first vector field to orbits of the second vector field in such a way that the homeomorphism preserves orientation but not necessarily parametrization by time. If h does preserve parametrization by time, then the dynamics generated by those vector field are said to be *conjugate.* Let us consider a vector field depending on one parameter, i.e., $\dot{x} = f_\mu(x)$, where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. It is said that the vector field undergoes a bifurcation at $\mu = \mu_0$ if vector fields $\dot{x} = f_{\mu<\mu_0}(x)$ are not conjugate with vector fields $\dot{x} = f_{\mu>\mu_0}(x)$. It is also said that the vector field $\dot{x} = f_{\mu_0}(x)$ is *structurally stable* if there is $\epsilon > 0$ such that vector fields $\dot{x} = f_\mu(x)$ are conjugate with $\dot{x} = f_{\mu_0}(x)$ if $|\mu - \mu_0| < \epsilon$. Here we define the notions of conjugacy and structural stability in terms of one parameter but we can extend these definitions to the vector field having more than one parameter. We remark that conjugacies do not need to be defined on all $\mathbb{R}^n$ but, rather, on appropriately chosen open sets in $\mathbb{R}^n$ especially an open set near the fixed point. It has been proven that the dynamics near a hyperbolic fixed point is structurally stable [47, chapter 1].

1.3.4 Center manifold theorem

In this section, we discuss one of the techniques necessary for the analysis of bifurcation problems. We will discuss the vector field having a non-hyperbolic fixed point, i.e., the matrix $Df(x_0)$ has a zero real-part eigenvalue. We first introduce the following theorem,

Theorem 1.2. *Let $\dot{x} = f(x)$ be a vector field on $\mathbb{R}^n$, x_0 is an equilibrium and let $A = Df(x_0)$. Divide the spectrum of A into three parts, σ_s, σ_u and σ_c with,*

$$\mathrm{Re}\ \lambda \begin{cases} < 0, & \text{if } \lambda \in \sigma_s; \\ = 0, & \text{if } \lambda \in \sigma_c; \\ > 0, & \text{if } \lambda \in \sigma_u. \end{cases} \tag{1.28}$$

Let the eigenspaces of σ_s, σ_u and σ_c be E^s, E^u and E^c respectively. Then there exist stable and unstable invariant manifolds W^s and W^u tangent to E^s and E^u at x_0 and a center manifold W^c tangent to E^c at x_0. The stable and unstable manifolds are unique but W^c does not need to be.

For more information on the existence, uniqueness and smoothness of these invariant manifolds and for a proof of this theorem, we refer to books on dynamical systems theory [47, 126] and references therein.

Without loss of generality, we assume that x_0 is the origin, i.e. $(0, 0, 0) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$. This theorem implies that the system having a non-hyperbolic fixed point, x_0, is locally topologically equivalent with,

$$\begin{aligned} \dot{x}_c &= A_c x_c + f_c(x_c, x_s, x_u), \\ \dot{x}_s &= A_s x_s + f_s(x_c, x_s, x_u), \\ \dot{x}_u &= A_u x_u + f_u(x_c, x_s, x_u) \qquad (x_c, x_s, x_u) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u, \end{aligned} \tag{1.29}$$

where

$$\begin{aligned} f_c(0,0,0) &= 0, \quad Df_c(0,0,0) = 0, \\ f_s(0,0,0) &= 0, \quad Df_s(0,0,0) = 0, \\ f_u(0,0,0) &= 0, \quad Df_u(0,0,0) = 0. \end{aligned} \tag{1.30}$$

In the above, A_c is a $c \times c$ matrix having eigenvalues with zero real parts, A_s and A_u are $s \times s$ and $u \times u$ matrices having eigenvalues with negative and positive real parts respectively. The case of interest is when the space σ_u is empty, hence we are left with these following equations,

$$\begin{aligned} \dot{x}_c &= A_c x_c + f_c(x_c, x_s), \\ \dot{x}_s &= A_s x_s + f_s(x_c, x_s), \qquad (x_c, x_s) \in \mathbb{R}^c \times \mathbb{R}^s. \end{aligned} \tag{1.31}$$

Since the center manifold is tangent to E^c (the space $x_s = 0$), then we can represent it as a (local) graph

$$W^c = \{(x_c, x_s) | x_s = h(x_c)\}, \quad h(0) = Dh(0) = 0, \tag{1.32}$$

where h is defined in some small neighbourhood $U \subset \mathbb{R}^c$ of the origin. The dynamics of the vector field restricted to the center manifold is,

$$\dot{x}_c = A_c x_c + f_c(x_c, h(x_c)). \tag{1.33}$$

The following theorem by Henry and Carr [47] that describes the dynamics of the vector field restricted to the center manifold (1.33) locally near the origin provides a good approximation to the flow (1.31) near the origin.

Theorem 1.3. *If the origin of (1.33) is locally asymptotically stable (unstable) then the origin of (1.31) is also asymptotically stable (unstable resp.).*

However, if we include the unstable direction (the space σ_u), the theorem above does not apply. It is still useful, though, to consider the dynamics restricted to the center manifold as it can be used to start our bifurcation analysis regarding the degeneracy in the linear part of the matrix $Df(x_0)$ (i.e. having a zero real part eigenvalue). In the end, we have to consider the inclusion of unstable directions that is important as it may be possible for such vector fields to undergo a secondary degeneracy bifurcation.

1.3.5 Normalization

In this section, we continue the development of technical tools which provide the basis for our study of the flow near a degenerate fixed point. We assume that the center manifold theorem has been applied to the system and henceforth we restrict our attention to the flow within the center manifold.

The method of normal form provides a way of finding a coordinate system in which the dynamical system takes the "simplest" form (the meaning of simplest is obviously contextual and will be explained as we go along). Three characteristics of this method should be mentioned.

1. The method is local in the sense that it holds in a sufficiently small neighbourhood of the fixed point.

2. If the vector field has a hyperbolic fixed point, then the normal form will just be the linearized system of the vector field about the hyperbolic fixed point.

3. The coordinate transformation will be non-linear functions of dependent variables.

Since the center manifold theorem has been applied we can consider a vector field

$$\dot{x} = Jx + f(x), \quad x \in \mathbb{R}^n$$

where the matrix J has eigenvalues with zero-real parts and the function f contains nonlinear terms of this vector field. We would like to find a coordinate change $x = h(y)$ with $h(0) = 0$ such that this vector field becomes the simplest possible. First we Taylor expand $f(x)$ so that the original vector field becomes:

$$\dot{x} = Jx + f_2(x) + f_3(x) + \ldots + f_{r-1}(x) + (O)(\|x\|^r), \tag{1.34}$$

where f_i represents the order i terms in the Taylor expansion of $f(x)$. In this normal form procedure we simplify (remove) the higher order terms f_i successively for $i = 2, 3, \ldots, r-1$. Firstly, we shall simplify the second order term, f_2, by introducing the following coordinate transformation,

$$x = h(y) = y + h_2(y), \tag{1.35}$$

where h_2 is second order in y. Substituting (1.35) into (1.34) gives,

$$\dot{x} = (I + Dh_2(y))\dot{y} = Jy + Jh_2(y) + f_2(y + h_2(y)) + f_3(y + h_2(y)) + \ldots + f_{r-1}(y + h_2(y)) + (O)(\|y\|^r), \tag{1.36}$$

where I denotes the $n \times n$ identity matrix. Note that for y sufficiently small, each term $f_j(y + h_2(y))$, $j = 2, \ldots, r-1$ can be written as

$$f_j(y + h_2(y)) = f_j(y) + \mathcal{O}(\|y\|^{j+1}) + \ldots + \mathcal{O}(\|y\|^{2j}), \tag{1.37}$$

and the inverse of $(I + Dh_2(y))$ exists and can be represented as a series expansion as follows,

$$(I + Dh_2(y))^{-1} = I - Dh_2(y) + \mathcal{O}(\|y\|^2). \tag{1.38}$$

Substituting (1.37) and (1.38) into (1.36) gives,

$$\dot{y} = Jy + (Jh_2(y) - Dh_2(y)Jy + f_2(y)) + \tilde{f}_3(y) + \ldots + \tilde{f}_{r-1}(y) + (O)(\|y\|^r). \tag{1.39}$$

Up to this point, h_2 is completely arbitrary. However, we now can choose a specific form of h_2 so as to simplify the $\mathcal{O}(\|y\|^2)$ terms as much as possible.

We firstly define the linear space of vector valued homogeneous polynomials of degree k, $H_k(\mathbb{R}^n)$. Then the linear term of the original vector field, $L = Jy$ induces a linear map that goes from this space to itself, defined as follows,

$$\begin{array}{rccc} \mathrm{ad}\, L: & H_k(\mathbb{R}^n) & \rightarrow & H_k(\mathbb{R}^n) \\ & h_k & \mapsto & [h_k, L], \end{array} \tag{1.40}$$

where $[h_k, L] = DLh_k(y) - Dh_k(y)L = Jh_k(y) - Dh_k(y)Jy$ that is actually the so-called Lie bracket operation. The linear space $H_2(\mathbb{R}^n)$ can be represented as

$$H_2(\mathbb{R}^n) = \text{ad } L(H_2(\mathbb{R}^n)) \oplus G_2,$$

where G_2 is a complement of the range of the ad L operator, ad $L(H_2(\mathbb{R}^n))$ in $H_2(\mathbb{R}^n)$. We go back to our original problem which is simplifying (1.39). It is clear that $f_2(y)$ can be viewed as an element in $H_2(\mathbb{R}^n)$ and so that $f_2(y)$ can be represented as follows,

$$f_2(y) = f_2^h(y) + f_2^g(y),$$

where $f_2^h(y)$ is in ad $L(H_k(\mathbb{R}^n))$ and $f_2^g(y)$ is the remaining part in G_2. Thus, (1.39) can be simplified to,

$$\dot{y} = Jy + f_2^g(y) + \tilde{f}_3(y) + \ldots + \tilde{f}_{r-1}(y) + (O)(\|y\|^r). \tag{1.41}$$

The term with a superscript g cannot be removed by the normalization, in other dynamical systems textbooks sometimes it is referred to as a "resonance" term. We can continue simplifying all the non-linear terms, i.e. $f_j, j = 3, \ldots, r-1$ to get,

$$\dot{y} = Jy + f_2^g(y) + f_3^g(y) + \ldots + f_{r-1}^g(y) + (O)(\|y\|^r). \tag{1.42}$$

At this point, the phrase "as simple as possible" should now become clear. When the vector field has a special structure such as symmetry etc., the normalization can still be performed. There will be restrictions, for instance when the vector field has a mirror symmetry (i.e. $y \mapsto -y$), then quadratic terms are not allowed. We conclude this section by saying that the center manifold reduction and normalization are two of the methods in dynamical systems to simplify a vector field. There are, of course, other methods, but we only use these two methods in this thesis.

1.3.6 Blowing up methods

When one is faced with a vector field whose linearization at some fixed point x_0 is hyperbolic, one can use the theorem (1.1) by Hartman and Grobman to determine the local phase portrait. However, when the point x_0 is non-hyperbolic, the theorem does not apply and we must include higher order terms. In the previous section, we have seen that the number of such non-linear terms can be reduced by using a normalization technique. The question we address is how we determine the local phase portrait and how far need such an expansion of normal forms go to determine the local vector field up to homeomorphism, just like the theorem (1.1).

If the vector field is one-dimensional, it might not be so hard to determine such a local vector field as it is determined by the lowest non-linear terms in the vector field. However, if the dimension of the vector field is higher than one, we really have a problem. So, the main tool for this problem is a technique that is the so-called *blowing up* technique. Singular changes of coordinates are introduced which expand a non-hyperbolic fixed point into a curve on which a number of equilibria occur. In this section, we are going to present an example of polar blowing-up in $\mathbb{R}^2$.

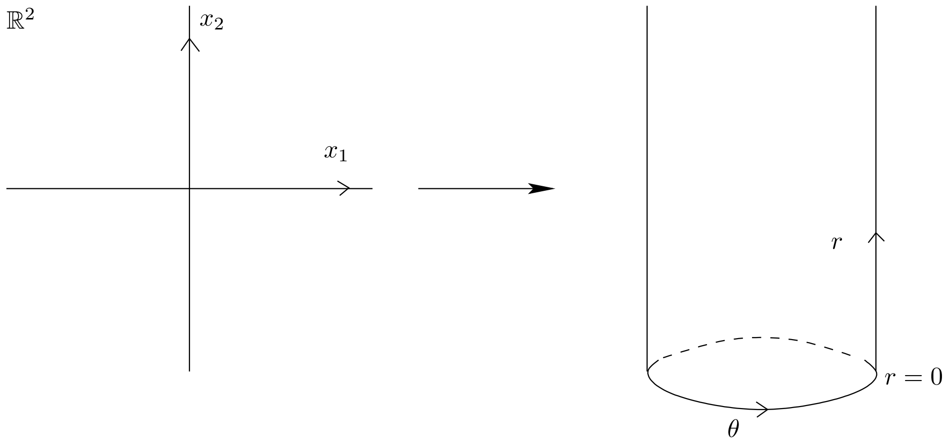

Figure 1.4: The polar blowing up by the transformation (1.43).

Suppose we have a vector field $\dot{x} = f(x) \in \mathbb{R}^2$. We introduce these transformations:

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{1.43}$$

to get a new vector field in terms of r and θ,

$$\dot{r} = rR(r,\theta) \quad \text{and} \quad \dot{\theta} = \Theta(r,\theta). \tag{1.44}$$

We are interested in the origin $(x_1, x_2) = (0, 0)$, which is brought to a circle $r = 0$ since the original phase plane $\mathbb{R}^2$ is now the upper half of a cylinder, while the origin becomes a circle $r = 0$, (see Figure 1.4). In the circle $r = 0$, there will be more than one equilibrium. If they are hyperbolic then we can determine the flow near those hyperbolic equilibria and we do a blowing down transformation to get the original flow of our vector field. If they are not hyperbolic, we can perform another blowing up to each equilibrium on this circle. This process is called successive blowing-up.

Besides the polar blowing-up, there are also different blowing up methods such as directional blowing ups. The following transformations describe examples of directional blowing ups in the x and y directions respectively,

$$x_1 = \bar{x}_1\bar{x}_2, \qquad x_2 = \bar{x}_2, \tag{1.45}$$

and

$$x_1 = \bar{x}_1, \qquad x_2 = \bar{x}_1\bar{x}_2. \tag{1.46}$$

Finally, these blowing up methods are used to find the local flow near a non-hyperbolic equilibrium. The next task is to consider the variation of parameters in our system that will be discussed in the next section.

1.3.7 Bifurcation theory

In bifurcation theory, we are interested in studying parameterized families of dynamical systems. However hyperbolic dynamics is insensitive to small changes (of parameters). Bifurcation theory will then focus on non-hyperbolic dynamics (equilibrium), in particular, the flow on the center manifold. When we want to study such a non-hyperbolic dynamical system, we will vary some parameters. This process is called an "unfolding" of bifurcations. The following questions are addressed.

1. What kind of local bifurcations do we typically have in dynamical systems with parameters?
2. Can we represent these local bifurcations by a normal form (simple model) with parameters?
3. If we vary some parameters of a particular dynamical system, will this dynamical system display the same bifurcation?

The answers to the above questions have been long discussed by various people. We will not go through the details, instead, we will summarize what has been done.

When a dynamical system has a degeneracy (i.e. a vector field has a non-hyperbolic equilibrium) we can use the center manifold theorem and a normalization technique to simplify the vector field locally. The local flow of such a degenerate vector field can also be determined by a blowing-up technique. Then when this normal form is varied by a parameter (i.e. an unfolding of bifurcation), we will have a bifurcation. How we vary parameters and what typical bifurcations occur has already been derived. We summarize all common local bifurcations and their unfolding of normal forms in the following.

1.3.7.1 Saddle-node bifurcation

Consider an autonomous dynamical system $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}^n$, depending on one parameter $\mu \in \mathbb{R}$, where f is smooth. Suppose at $\mu = 0$ the vector field has the origin as the equilibrium and the linearized matrix of the vector field evaluated at the origin, $f_x(0, 0)$, has a single zero eigenvalue. Since the center manifold of this degeneracy is one-dimensional, we can reduce the system to a one-dimensional vector field $\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \mu)$, $\tilde{x}, \mu \in \mathbb{R}^1$. Suppose the following non-degeneracy conditions also hold:

1. $\dfrac{d\tilde{f}}{d\mu}(0, 0) \neq 0$, and
2. $\dfrac{d^2\tilde{f}}{d\tilde{x}^2}(0, 0) \neq 0$.

In Kuznetsov [78] the first condition above is called a transversality condition while the second condition above is called a non-degeneracy condition. As μ passes through $\mu = 0$, a saddle-node bifurcation occurs. Moreover, the vector field is topologically equivalent near $(0, 0)$ to the following normal form,

$$\dot{y} = \alpha \pm y^2, \qquad y, \alpha \in \mathbb{R}, \tag{1.47}$$

at which two equilibria appear and vanish. Here, the normal form is one-dimensional as the center manifold of the single zero eigenvalue degeneracy is one-dimensional. This bifurcation is well known and all information about this bifurcation can be found in any bifurcation text book [126].

We remark that this bifurcation is codimension-one since there is only one degeneracy which is the single zero eigenvalue. Generally, we only need one parameter in order to violate this degeneracy. Therefore, a vector field is possible to undergo a codimension-one bifurcation by varying only one parameter.

1.3.7.2 Transcritical bifurcation

Consider an autonomous dynamical system $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$, depending on one parameter $\mu \in \mathbb{R}$, where f is smooth. Suppose at $\mu = 0$ the vector field has the origin as an equilibrium and the linearized matrix of the vector field evaluated at the origin, $f_x(0,0)$, has a single zero eigenvalue. As the center manifold is one-dimensional, we can reduce the system to a one-dimensional vector field $\dot{\tilde{x}} = \tilde{f}(\tilde{x},\mu)$, $\tilde{x}, \mu \in \mathbb{R}^1$ and suppose the first non-degeneracy condition in the saddle-node bifurcation is violated, i.e., $\frac{d\tilde{f}}{d\mu}(0,0) = 0$. Lastly, suppose the following non-degeneracy conditions hold:

1. $\dfrac{d^2\tilde{f}}{d\tilde{x}d\mu}(0,0) \neq 0$, and
2. $\dfrac{d^2\tilde{f}}{d\tilde{x}^2}(0,0) \neq 0$.

Then as μ passes through $\mu = 0$, a transcritical bifurcation occurs. Moreover, the vector field is topologically equivalent near $(0,0)$ to the following normal form,

$$\dot{y} = \alpha y \pm y^2, \qquad y, \alpha \in \mathbb{R}, \tag{1.48}$$

at which an equilibrium crosses another equilibrium and exchanges stabilities. Here, the normal form is one-dimensional as the center manifold of the single zero eigenvalue degeneracy is one-dimensional. This bifurcation is not a typical codimension-one bifurcation. It is mainly because there are two degeneracy conditions we have to meet while there is only a single parameter to vary. This bifurcation occurs mainly when one varies the parameter in such a way that the non-degeneracy condition (of a saddle-node bifurcation) is violated. This bifurcation might also occur in a system having a special structure so that we always have the non-degeneracy condition violated.

1.3.7.3 Hopf bifurcation

Consider an autonomous dynamical system $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$, depending on one parameter $\mu \in \mathbb{R}$, where f is smooth. Suppose the following conditions hold:

1. for all sufficiently small $|\mu|$ the system has a family of equilibria, $x_0(\mu)$,
2. the first derivative of the vector field, $f_x(x_0(\mu),\mu)$ has one pair of complex eigenvalues

   $$\lambda(\mu) = \mathrm{Re}\ (\lambda(\mu)) + i\mathrm{Im}\ (\lambda(\mu)),$$

   that becomes purely imaginary when $\mu = 0$, i.e., $\mathrm{Re}\ (\lambda(0)) = 0$ and $\mathrm{Im}\ (\lambda(0)) = \omega > 0$.

Suppose the following non-degeneracy conditions also hold:

1. $\dfrac{d\mathrm{Re}\ (\lambda)}{d\mu}(0) \neq 0$, and

2. $l_1(0) \neq 0$.

Then as μ passes through $\mu = 0$, a Hopf bifurcation occurs. Moreover, the vector field is topologically equivalent near $x_0(0)$ to the following normal form,

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 - \omega y_2 + (a y_1 - b y_2)(y_1^2 + y_2^2), \\ \dot{y}_2 &= \omega y_1 + \alpha y_2 + (b y_1 + a y_2)(y_1^2 + y_2^2). \end{aligned} \tag{1.49}$$

At which, $\alpha = 0$, the equilibrium changes stability and a unique limit cycle bifurcates from it. The function $l_1(\mu)$ is called the Lyapunov coefficient. This can be obtained when one computes the normal form of the vector field with a pair of purely imaginary eigenvalues. We will not describe how to obtain the Lyapunov coefficient, instead we refer to Kuznetsov [80] for more information how to compute this coefficient from the normal form derivation. Here, the normal form is two-dimensional as the center manifold of this degeneracy is two-dimensional. This bifurcation is also well known in the field of bifurcation theory. Remark that the Hopf bifurcation is also codimension-one as there is only one degeneracy (i.e. a pair of purely imaginary eigenvalues).

The saddle-node and Hopf bifurcations are known to be codimension-one bifurcations as they have one degeneracy. Additional codimension-one bifurcations that a vector field could undergo are pitchfork bifurcation (i.e. a single zero eigenvalue degeneracy in a vector field with $\mathbb{Z}_2$ symmetry), homoclinic bifurcation (i.e. an occurrence of homolinic solutions) and heteroclinic bifurcation (i.e. an occurrence of heteroclinic solution). More degeneracies that could come from the lowest non-linear term in the normal form and the linear part of the vector field could also occur. These give rise to more degenerate bifurcations, namely codimension two bifurcations. These are common codimension two bifurcations; cusp, Bautin, Bogdanov-Takens and fold-Hopf bifurcations. The unfoldings of such codimension two bifurcations have also been studied. We can now say that if we perform an unfolding of a vector field and we find these typical bifurcations (codimension-one and codimension two bifurcations) then the unfolding is *versal* (or *universal*). One characteristic of a versal unfolding is when the transformation that brings the parameter space of the original vector field to the parameter space of the normal form is invertible.

This bifurcation analysis is usually represented by using a bifurcation diagram which plots bifurcation conditions with respect to parameters that we vary. In the two-dimensional parameter space bifurcation set, a codimension-one bifurcation will be a curve, and a codimension-two bifurcation will be a point, which is made up by an interaction of two or more codimension-one bifurcation curves.

Up to this point, we have listed all typical bifurcations and we have confirmed that these bifurcations can be represented by normal forms with their versal unfoldings. However, the third question raised at the start of section 1.3.7 has not been completely solved. This is also the question that we try to answer in this thesis.

1.3.8 Additional notes to references

The history of dynamical systems goes back to three centuries ago when Euler attempted to study the motion of the moon that is now so-called the N-body problem. Not only that, but there are also a lot of other problems in science and engineering that can be modelled as a dynamical system. To name a few, population dynamics, atmospheric models, the prediction of stock market prices are among those problems. A revolutionary contribution to the theory of dynamical systems is due to Henri Poincaré. His proposal is to look at the geometry of solutions instead of the explicit formula of solutions. For an introduction to the modern theory of dynamical systems we refer to Katok and Hasselblatt [66]. For a mathematically oriented approach to dynamical systems theory, we refer to Guckenheimer and Holmes [47] and Wiggins [126]. The theory of center manifold and normalization technique are two of the modern developments in the theory of dynamical systems. More information can be found in Broer et al. [12]. Bifurcation theory is another subject in the modern theory of dynamical systems which finds its origin in the work of Poincaré. For a more practical approach to bifurcation theory, we refer to Kuznetsov [78]. The blowing-up method is also one of the new developments of the theory of dynamical systems, (see the paper by Takens [116]).

1.4 Outline of this thesis

Since this thesis is a collection of studies that analyse the Lotka-Volterra system with constant terms, each chapter can be read separately. In chapter 2, we will discuss bifurcation analysis of two-dimensional Lotka-Volterra systems with a constant term. It is shown that the Lotka-Volterra system with a constant term undergoes a series of bifurcations when we vary the constant term. Moreover, if we also vary another parameter, it is shown that more degenerate bifurcations can occur. We show that these bifurcations are different from typical bifurcations that we usually find in text books. In this chapter, we analyse those unusual degenerate bifurcations, and explain how these bifurcation are created and unfolded in our system.

The analysis in chapter 2 does not deal with why these unusual bifurcations are there in the first place. One of the reasons this happens is because of a special structure that is possessed by our system. In chapter 3, we shall perform an analysis of a dynamical system that possesses the same special structure as our system does, namely a codimension-one invariant manifold that exists for every value of the parameters. We will perform a general global bifurcation analysis on a general system with an invariant manifold and show that the system with an invariant manifold will undergo a series of different bifurcations which do not normally occur in a general system. We also show that the Lotka-Volterra system with a constant term undergoes the same bifurcations that are found in this chapter.

In chapter 4, we discuss the existence of first integrals of dynamical systems, in particular of two and three-dimensional Lotka-Volterra systems with constant terms. The existence of first integrals of two and three-dimensional Lotka-Volterra equations has already attracted much attention, however, we believe that the first integral of Lotka-Volterra systems with constant terms has not been analysed. We will discuss the cases when all constant terms are not zero, at least one constant term is zero, respectively when all of the constant terms are zero. The latter is included in our thesis in order to have a complete investigation of all possible cases. It is shown that the Lotka-Volterra system with constant terms is integrable for specific values of the parameters.

In chapter 5, a discussion of the major results presented in this thesis is given. We discuss the common thread that connects all of our results together. We also discuss some ideas and possible future projects we can pursue regarding results obtained in chapters 2, 3 and 4. Finally, an overall conclusion is given.

CHAPTER 2

Unusual bifurcations in the Lotka-Volterra system with a constant term

2.1 Introduction

Interactions between bifurcations of equilibria and of cycles (i.e. periodic orbits) occur naturally in dynamical systems. In the bifurcation diagram, the interaction points often act as organising centres, at which local and global bifurcations converge and the behaviour of the system is determined to a large extent. These interaction points have been the subject of intensive research over the last decades and, in particular, all interaction points which occur in general dynamical systems have been classified and analysed in the literature, see for instance, Kuznetsov [78].

However, the theory of interactions between bifurcations for systems with a special structure is, as yet, incomplete. In such systems, bifurcations can have a lower codimension than that in the general case. For instance, the presence of a Z^2 symmetry in the dynamical system can render the pitchfork bifurcation codimension-one, whereas the number of degeneracies of this bifurcation is the same as that of general codimension-two bifurcations. Pitchfork bifurcations occur in particular in certain normal forms with a special structure namely S^1 symmetry, such as the saddle-node–Hopf normal form, after decoupling of the angular variable, see Guckenheimer and Holmes [47]. For this reason interactions with pitchfork bifurcations have been extensively investigated. Here we mention a few contributions: Scheurle and Marsden [109] discussed the existence of tori and quasiperiodic flows resulting from saddle-node–Hopf bifurcations, while Broer and Vegter [13] discussed the existence of Shilnikov bifurcations. The existence of heteroclinic orbits was investigated by Lamb *et al.* [84] for the saddle-node–Hopf system with time reversal symmetry and for the saddle-node–pitchfork system by Kirk and Knobloch [70].

Interactions that involve a transcritical bifurcation, in contrast, have not attracted much attention. This might be because this bifurcation does not correspond to a global phase space symmetry, like the pitchfork bifurcation. However, the transcritical bifurcation is generic in dynamical systems in which an equilibrium must exist for all values of the parameters. An example with this property comes from population dynamic models. In the Lotka-Volterra type model, the variables are the population densities of several species. If a species dies out, it cannot be regenerated and therefore the coordinate axes of such a model are invariant and the origin is always an equilibrium solution (see, e.g. Zhu et al. [131]). In this current chapter, we will investigate the interactions between saddle-node and transcritical bifurcations that occur in the two-dimensional Lotka-Volterra system with a constant term. To our best knowledge, this interaction has not been reported on in the literature before.

From a bifurcation theory point of view, the transcritical bifurcation can be considered as a non-versal unfolding of the well-know saddle-node bifurcation. The normal form of the saddle-node bifurcation is given by

$$\dot{x} = \mu + x^2,$$

while the normal form of the transcritical bifurcation is given by

$$\dot{y} = \alpha y + y^2.$$

If we apply the transformation $z = y + \frac{\alpha}{2}$ to the system above we obtain

$$\dot{z} = -\frac{\alpha^2}{4} + z^2,$$

which is a normal form of the saddle-node bifurcation parameterized by α. Thus, we can consider the transcritical bifurcation as an unfolding of the saddle-node bifurcation. Because the map $\mu = -\alpha^2/4$ is noninvertible at the bifurcation point $\alpha = \mu = 0$, this unfolding is non-versal.

Using the idea above, we investigate two different saddle-node–transcritical interactions, which correspond to a single zero and a double zero eigenvalue degeneracy respectively. In the former case, there is no additional bifurcation and the bifurcation diagram around the interaction can be considered as a non-versal unfolding of the cusp bifurcation. The second case is more complicated as the normal form of a double zero eigenvalue degeneracy is the Bogdanov-Takens (BT) normal form. Due to nondegeneracy conditions of the transcritical bifurcation, we obtain the normal form of a *degenerate* BT (DBT) bifurcation. Not only the saddle-node and transcritical bifurcations are present, but there are also Hopf, homoclinic and heteroclinic bifurcations appearing. We find two topologically different diagrams corresponding to different unfoldings of the DBT singularity, namely, the *elliptic* and the *saddle* cases. A complete topological classification of the DBT bifurcation has been analysed in Dumortier et al. [31].

2.2 Bifurcation diagram of the Lotka-Volterra system with a constant term

The two-dimensional Lotka-Volterra model with a constant term is given below,

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2) + e, \\ \dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2), \end{aligned} \tag{2.1}$$

where the constant e can be thought of as a constant rate harvesting or a migration term [11, 127]. Without the constant term, the origin is always an equilibrium and both the x-axis and the y-axis are invariant. In the presence of the constant term, this equilibrium is moved along the x-axis, which is still invariant. The system (2.1) has at most four equilibria depending on parameters. Two equilibria,

$$\begin{aligned} R_1 &= (\frac{-b_1 + \sqrt{{b_1}^2 - 4a_{11}e}}{2a_{11}}, 0) \\ R_2 &= (\frac{-b_1 - \sqrt{{b_1}^2 - 4a_{11}e}}{2a_{11}}, 0) \end{aligned} \tag{2.2}$$

are sitting on the x-axis while the last two equilibria are

$$\begin{aligned} R_3 &= (\rho^+, \frac{-b_2 - a_{21}\rho^+}{a_{22}}) \\ R_4 &= (\rho^-, \frac{-b_2 - a_{21}\rho^-}{a_{22}}) \end{aligned} \tag{2.3}$$

where

$$\rho^{+,-} = \frac{-(-b_1a_{22} + b_2a_{12}) \pm \sqrt{(-b_1a_{22} + b_2a_{12})^{\ 2} - 4ea_{22}D_1}}{2D_1},$$

with $D_1 = a_{11}a_{22} - a_{12}a_{21}$. The coordinates and the seven parameters are related by three continuous symmetries:

$$\begin{aligned} (x_1, a_{11}, a_{21}, e) &\mapsto (\lambda x_1, \tfrac{1}{\lambda}a_{11}, \tfrac{1}{\lambda}a_{21}, \lambda e) \\ (x_2, a_{12}, a_{22}) &\mapsto (\mu x_2, \tfrac{1}{\mu}a_{12}, \tfrac{1}{\mu}a_{22}) \\ (b_1, b_2, a_{11}, a_{21}, a_{12}, a_{22}, e, t) &\mapsto (\kappa b_1, \kappa b_2, \kappa a_{11}, \kappa a_{21}, \kappa a_{12}, \kappa a_{22}, \kappa e, \tfrac{1}{\kappa}t) \end{aligned} \tag{2.4}$$

for any $\lambda, \mu, \kappa \neq 0$. In this chapter, we will use b_2 and e as bifurcation parameters, fix a_{11} and a_{12} to distinguish the topologically different bifurcation diagrams (i.e. saddle, $a_{11} = -5, a_{12} = -3$ and elliptic, $a_{11} = 7, a_{12} = -3$). The topological difference will be explained later in this chapter. We also fix the remaining parameters as $b_1 = 15$, $a_{21} = 2$ and $a_{22} = -1$.

When varying the parameter e, the system (2.1) undergoes a series of codimension-one bifurcations. Equilibria R_1 and R_2 collide in a saddle-node bifurcation when,

$$e = \frac{{b_1}^2}{4a_{11}}. \tag{2.5}$$

This is the first saddle-node bifurcation since equilibria R_3 and R_4 might coalesce in the second saddle-node bifurcation when

$$e = \frac{(-b_1a_{22} + b_2a_{12})^2}{4a_{22}D_1}. \tag{2.6}$$

Also, either of the equilibria R_3 and R_4 will cross the x-axis through one of the equilibria R_1 and R_2 and exchange stability in a transcritical bifurcation when

$$e = \frac{b_2(-b_2a_{11} + b_1a_{21})}{{a_{21}}^2}. \tag{2.7}$$

Those are codimension-one bifurcations. We prove the degeneracy and the non-degeneracy conditions of the above bifurcations in the Appendix B. We can vary the other bifurcation parameter, b_2 to have more degenerate bifurcations (i.e. codimension-two bifurcations). There are at least two codimension-two bifurcations that we find when varying e and b_2. The first

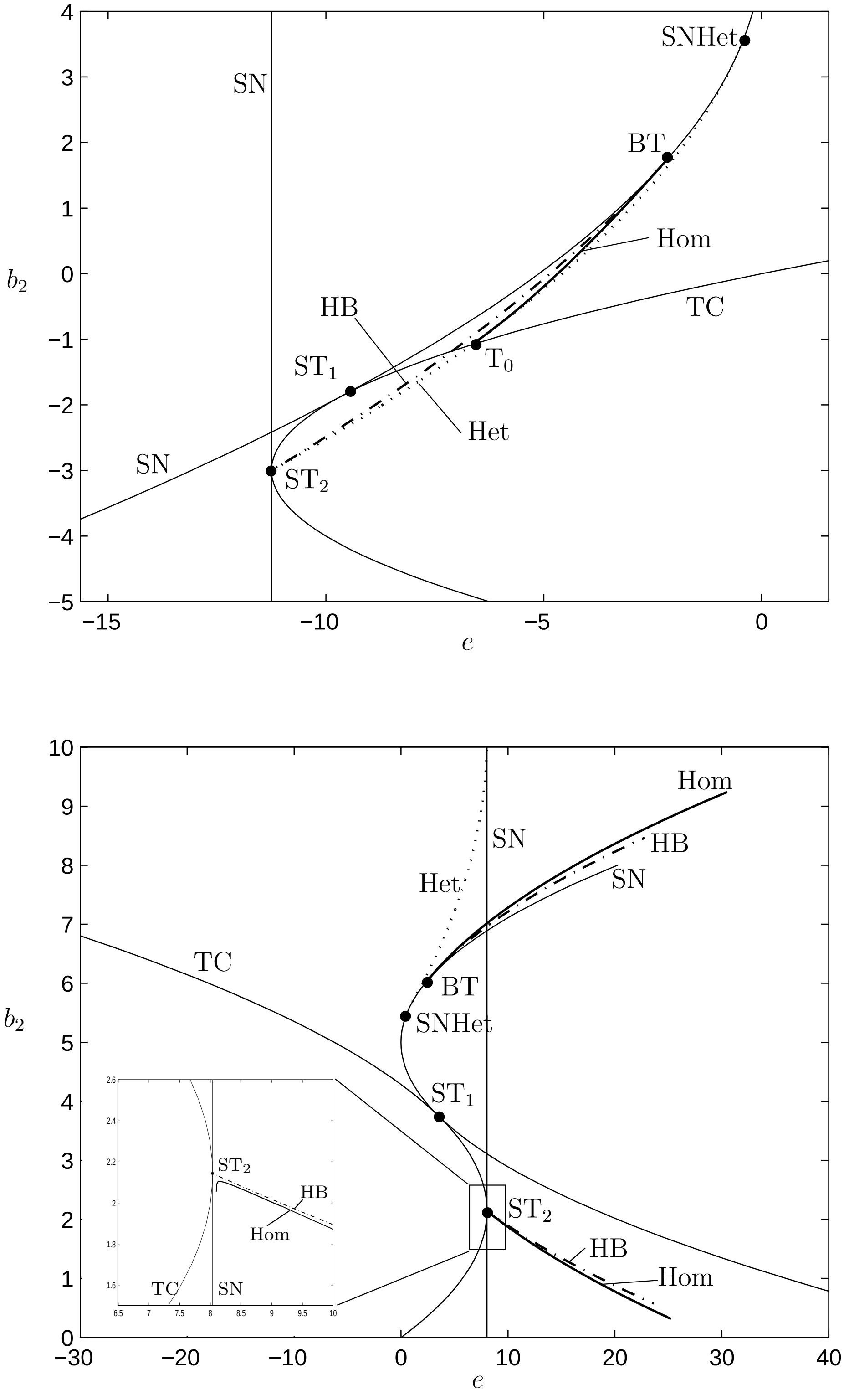

Figure 2.1: Bifurcation diagrams of (2.1) with e and b_2 as parameters. The two saddle-node–transcritical interactions have been marked ST_1 and ST_2. Top: the *saddle* case for ST_2, with $b_1 = 15$, $a_{11} = -5$, $a_{12} = -3$, $a_{21} = 2$, $a_{22} = -1$. Bottom: the *elliptic* case for ST_2, with $a_{11} = 7$ and all other parameters as in the saddle case. All the labels are explained in Table 2.1.

Symbols	Bifurcations	Codimension of bifurcation
BT	Bogdanov-Takens	2
HB	Hopf	1
Het	Heteroclinic	1
Hom	Homoclinic	1
SN	Saddle-node	1
ST	Saddle-node–transcritical	2
TC	Transcritical	1
T_0	Heteroclinic–homoclinic	2

Table 2.1: List of bifurcations that occur in the Lotka Volterra system with a constant term.

codimension-two bifurcation occurs when the system satisfies the conditions of the second saddle-node bifurcation (2.6) and of the transcritical bifurcation (2.7), at which three equilibria (R_3, R_4 and R_1) collide. It turns out that if the parameters satisfy these conditions, the linearized matrix of the system (2.1),

$$J(R_1 = R_3 = R_4) = \begin{pmatrix} -\dfrac{b_1 a_{12} a_{21}}{D_2} & -\dfrac{b_1 a_{12} a_{22}}{D_2} \\ 0 & 0 \end{pmatrix} \tag{2.8}$$

has a single-zero eigenvalue.

The second codimension-two bifurcation that we have found is when the parameters satisfy conditions of the first saddle-node bifurcation (2.5) and of the transcritical bifurcation (2.7), at which three equilibria (R_1, R_2 and R_3) collide. The linearized matrix of the system is,

$$J(R_1 = R_2 = R_3) = \begin{pmatrix} 0 & -\dfrac{b_1 a_{12}}{2a_{11}} \\ 0 & 0 \end{pmatrix}. \tag{2.9}$$

In Figure 2.1 a complete bifurcation diagram is presented for two topologically different cases. In each bifurcation diagram, there are two saddle-node bifurcations (labelled SN) and one transcritical bifurcation (labelled TC). The first saddle-node bifurcation, which is represented by a vertical line in both figures, is a collision between equilibria that lie on the x-axis (R_1 and R_2). The other saddle-node bifurcation curve involves the other two equilibria (R_3 and R_4). The transcritical bifurcation coincides with the second saddle-node bifurcation to create the single zero eigenvalue degeneracy (labelled ST_1), while the double zero eigenvalue degeneracy (labelled ST_2) is created from the interaction of the transcritical bifurcation with the first saddle-node bifurcation. Additional codimension-one bifurcations also appear such as a Hopf bifurcation curve (labelled HB), heteroclinic connections (labelled Het) and homoclinic loops (labelled Hom). Unlike saddle-node and transcritical bifurcation, which are obtained analytically, these additional bifurcations are detected numerically by a continuation software package for dynamical systems, namely AUTO2000 (see Doedel et al. [29]). Continuing further those codimension-one bifurcation curves we obtain codimension-two bifurcation points such as a Bogdanov-Takens (BT), saddle-node/heteroclinic bifurcation

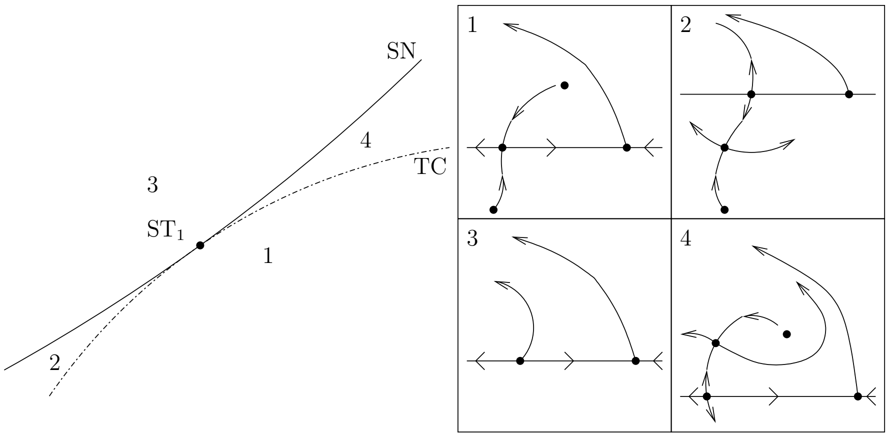

Figure 2.2: Phase portraits of the Lotka-Volterra system (2.1) around the single zero saddle-node–transcritical interaction ST_1. Labels are explained in Table 2.1.

(SNHet) and homoclinic/heteroclinic bifurcation(T_0). In this chapter, we are particularly interested in the interaction of the transcritical bifurcation and the saddle-node bifurcations. In Figure 2.1 we label the interaction of saddle-node and transcritical bifurcations by ST_1 and ST_2.

We will first discuss the single zero saddle-node–transcritical eigenvalue interaction and show that it is related to a non-versal unfolding of the codimension-two cusp bifurcation. Then turning to the double zero eigenvalue saddle-node–transcritical interaction, we will show how it is related to non-versal unfoldings of the DBT singularity.

2.3 The single zero saddle-node–transcritical interaction

In Figure 2.2, we see the dynamics around the first saddle-node–transcritical interaction ST_1. Three equilibria are involved in this interaction, one of which lies on the invariant axis (R_1) while the others (R_3 and R_4) are created in a saddle-node bifurcation.

2.3.1 The minimal model

A simple model that undergoes the same qualitative behaviour as the first interaction of the Lotka-Volterra model shown in Figure 2.2 is given by

$$\dot{x} = ax + bx^2 + \epsilon x^3, \tag{2.10}$$

where $\epsilon = \pm 1$. Note that we can restrict our analysis to the case $\epsilon = 1$, which is related to the case $\epsilon = -1$ through the transformation $(x, a, b, t, \epsilon) \to (-x, -a, b, -t, -\epsilon)$. Also, note that this is the normal form of the transcritical bifurcation that is extended to a third-order term. This model (with $\epsilon = 1$) has three equilibria:

$$\begin{aligned}
x_0 &= 0, \\
x_1 &= -\frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4a}, \\
x_2 &= -\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4a}.
\end{aligned}$$

We now want to find the value of parameters such that there is a degeneracy in the stability of the above equilibria. By computing the first derivative of the system (2.10) evaluated at those equilibria, we have the following statements:

1. the equilibrium x_0 is degenerate with a zero eigenvalue when $a = 0$,
2. the equilibrium x_1 (and also x_2) is degenerate with a zero eigenvalue when $a = 0$ and $b > 0$ ($a = 0$ and $b < 0$ respectively),
3. the equilibrium x_1 and x_2 are degenerate with a zero eigenvalue when $a = \frac{b^2}{4}$.

The equilibria x_1 and x_2 coalesce in a saddle-node bifurcation along $a = \frac{b^2}{4}$, and either of them crosses the equilibrium x_0 in a transcritical bifurcation along $a = 0$. We set $f(x, a) = ax + bx^2 + x^3$ to check the non-degeneracy conditions of the saddle-node bifurcation along $a = b^2/4$:

$$\frac{\partial f}{\partial a}(x_1, b^2/4) = \frac{\partial f}{\partial a}(x_2, b^2/4) = -\frac{b}{2} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x_1, b^2/4) = \frac{\partial^2 f}{\partial x^2}(x_2, b^2/4) = -b, \tag{2.11}$$

and those of the transcritical bifurcation along $a = 0$:

$$\frac{\partial f}{\partial a}(0,0) = 0, \quad \frac{\partial^2 f}{\partial a \partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 2b. \tag{2.12}$$

From the computations above, we can conclude that, in the plane of parameters a and b, a codimension-one saddle-node bifurcation takes place along the line $a = b^2/4$ and a transcritical bifurcation takes place along the line $a = 0$. The only point at which these bifurcations are degenerate is the origin, at which only one equilibrium exists.

2.3.2 Relation to the normal form of cusp bifurcation

The following translation

$$z = x + \frac{b}{3} \tag{2.13}$$

transforms the minimal model (2.10) with $\epsilon = 1$ into the standard unfolding of the cusp bifurcation

$$\dot{z} = \mu + \nu z + z^3, \tag{2.14}$$

with unfolding parameters μ and ν that are now functions of the parameters of the minimal model a and b:

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \phi(a, b) = \begin{pmatrix} -\frac{1}{3}ab + \frac{2}{27}b^3 \\ a - \frac{1}{3}b^2 \end{pmatrix} \tag{2.15}$$

Thus, we can consider the minimal model of this saddle-node–transcritical interaction as an unfolding of the cusp normal form. This unfolding is, however, nonversal because the map ϕ is non-invertible along part of the bifurcation set.

The bifurcation diagram of the cusp normal form has the well known Λ-shaped curve of saddle-node bifurcations, that is given by the following expression:

$$\frac{1}{4}\mu^2 + \frac{1}{27}\nu^3 = 0. \tag{2.16}$$

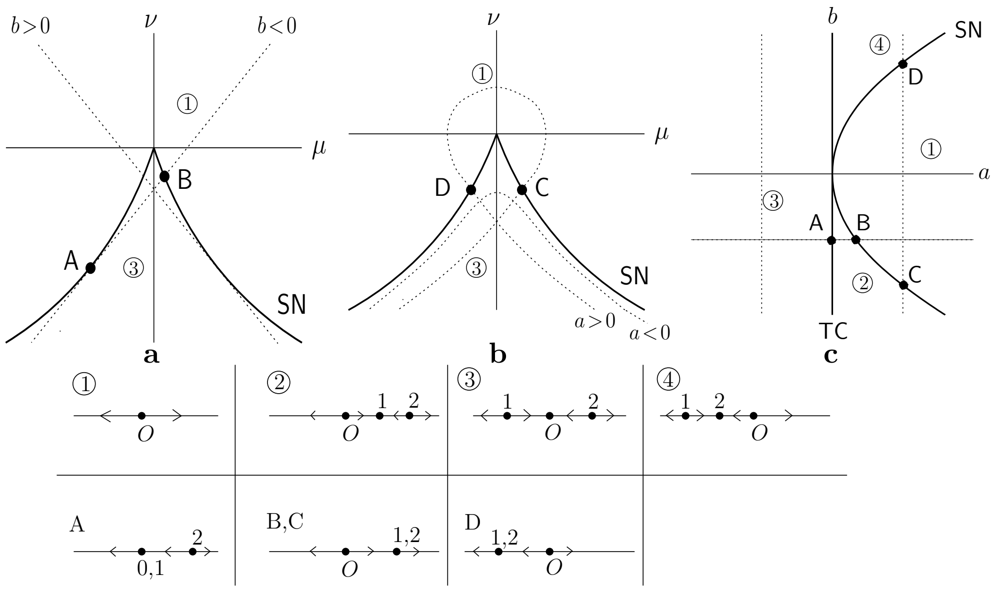

Figure 2.3: Illustration of the saddle-node–transcritical bifurcation as a nonversal unfolding of the cusp bifurcation. **a**: Unfolding (2.14) with the cusp bifurcation at the origin. The dotted curves denote isolines of positive and negative b in model (2.10). **b**: Likewise with isolines of positive and negative a. **c**: Bifurcation diagram of model (2.10) with the corresponding lines of constant a, b. The transversal intersections B, C and D correspond to saddle-node bifurcations (SN) whereas the tangency A corresponds to a transcritical bifurcation (TC).

The preimage of this set under the map ϕ consists of two components, given by

$$\begin{aligned} &a = \frac{1}{4}b^2, \text{ at which } \det(\mathrm{D}\phi) = \frac{1}{12}b^2 \text{ and} \\ &a = 0, \text{ at which } \det(\mathrm{D}\phi) = 0. \end{aligned} \tag{2.17}$$

If we exclude the codimension-two point at the origin, the map ϕ is invertible along the first component, corresponding to a saddle-node bifurcation. On the other hand, the Jacobian of this map has rank one when $a = 0$, which explains why this curve corresponds to the transcritical bifurcation, which is more degenerate. In Figure 2.3, the two bifurcation sets are depicted along with lines that are explained in the following. A line b is constant is mapped onto a straight line in the plane of parameters μ and ν. This line intersects the Λ-shaped bifurcation set twice, once tranversally and once in a tangency, which correspond respectively to a saddle-node bifurcation and a transcritical bifurcation in parameter space a and b. A line a is constant is mapped onto a curve which either has no intersection with the bifurcation set ($a < 0$) or has two transversal intersections ($a > 0$) or coincides with the bifurcation set ($a = 0$).

2.3.3 Equivalence to the Lotka-Volterra model with a constant term

The minimal model (2.10) is equivalent to the reduction of the Lotka-Volterra model with a constant term (2.1) to the one-dimensional centre manifold at the saddle-node–transcritical interaction ST_1. This codimension-two point is located at

$$\begin{aligned} b_2^* &= \frac{b_1 a_{22} a_{21}}{D_2} & x_1^* &= -\frac{b_2^*}{a_{21}}, \\ e^* &= \frac{b_1^2 a_{22} D_1}{D_2^2} & x_2^* &= 0, \end{aligned} \tag{2.18}$$

where we define,

$$D_1 = a_{11}a_{22} - a_{12}a_{21}, \qquad D_2 = 2a_{11}a_{22} - a_{12}a_{21}. \tag{2.19}$$

After an initial transformation given by

$$\begin{aligned} x_1 &= x_1^* + z_1 - \frac{a_{22}}{a_{21}} z_2 + z_3, & e &= e^* + \frac{b_1 a_{12} a_{21}}{D_2} z_3, \\ x_2 &= x_2^* + z_2, & b_2 &= b_2^* + z_4, \end{aligned} \tag{2.20}$$

the Lotka-Volterra system (2.1) can be written as the extended system

$$\begin{aligned} \dot z_1 &= -\frac{b_1 a_{12} a_{21}}{D_2} z_1 + a_{11} z_1^2 - \frac{D_3}{a_{21}} z_1 z_2 + \frac{a_{22}}{a_{21}^2} D_1 z_2^2 + a_{11} z_3^2 + 2a_{11} z_1 z_3 \\ &\qquad - \frac{D_3}{a_{21}} z_2 z_3 + \frac{a_{22}}{a_{21}} z_2 z_4, \\ \dot z_2 &= a_{21} z_1 z_2 + a_{21} z_2 z_3 + z_2 z_4, \\ \dot z_3 &= 0, \\ \dot z_4 &= 0, \end{aligned} \tag{2.21}$$

where

$$D_3 = 2a_{11}a_{22} - a_{12}a_{21} - a_{22}a_{21}. \tag{2.22}$$

This system has a three-dimensional center manifold which can be represented locally as the graph of a function $z_1 = \psi(z_2, z_3, z_4)$. The Taylor expansion of this function is found to be

$$\psi(z_2, z_3, z_4) = \frac{D_2}{b_1 a_{21} a_{12}} \left(\frac{a_{22} D_1}{a_{21}^2} z_2^2 + a_{11} z_3^2 - \frac{D_3}{a_{21}} z_2 z_3 + \frac{a_{22}}{a_{21}} z_2 z_4 \right) + \text{h. o. t.} \tag{2.23}$$

Thus, the dynamics in the centre manifold is the following one-dimensional system,

$$\dot z_2 = (z_4 + a_{21} z_3) z_2 + \frac{D_2 z_2}{b_1 a_{12}} \left(\frac{a_{22} D_1}{a_{21}^2} z_2^2 + a_{11} z_3^2 - \frac{D_3}{a_{21}} z_2 z_3 + \frac{a_{22}}{a_{21}} z_2 z_4 \right) + \text{h. o. t.} \tag{2.24}$$

Now if we scale the variable as follows,

$$x = \sqrt{\left| \frac{a_{22} D_1 D_2}{b_1 a_{12} a_{21}^2} \right|}\, z_2, \tag{2.25}$$

we find the minimal model (2.10) with

$$\begin{aligned} \epsilon &= \operatorname{sign}\left(\frac{a_{22} D_1 D_2}{b_1 a_{12}} \right), \\ a &= a_{21} z_3 + \frac{a_{11} D_2}{b_1 a_{12}} z_3^2 + z_4, \\ b &= \frac{\epsilon a_{21}}{d_1} \sqrt{\left| \frac{a_{22} D_1 D_2}{b_1 a_{12} a_{21}^2} \right|} (z_4 - \frac{D_3}{a_{22}} z_3). \end{aligned} \tag{2.26}$$

The last two relations define a map from the parameters z_3 and z_4 to the parameters a and b, which is smooth and invertible on an open neighbourhood of the codimension-two point $(z_3, z_4) = \left(\frac{D_2}{b_1 a_{12} a_{21}} (e - e^*), b - b^* \right) = (0, 0)$.

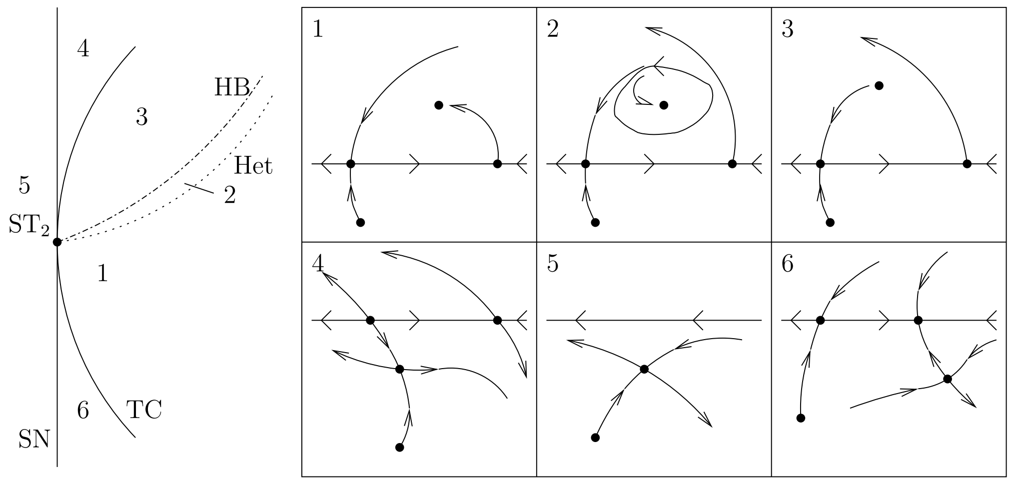

Figure 2.4: The dynamics around the second interaction of the Lotka-Volterra system for the saddle case. Labels are explained in Table 2.1.

2.4 The double zero saddle-node–transcritical interaction

In Figures 2.4 and 2.5 the bifurcations around the saddle-node–transcritical interactions with two zero eigenvalues are shown. Again, three equilibria are involved, but in this case, limit cycles and connecting orbits are generated.

2.4.1 The minimal model

A simple model to represent the double-zero interaction of saddle-node and transcritical bifurcation is given by

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= ax + k_1by + bx^2 + k_2xy + x^2y + \epsilon x^3,\end{aligned} \tag{2.27}$$

where $k_1, k_2 \neq 0$ and $\epsilon = \pm 1$. The bifurcation parameters are a and b. When these parameters are both zero, $a = b = 0$, the linearized matrix of the above system has a double-zero eigenvalues. When solving for equilibria, we find the same equation as for model (2.10) except that now we do not specify a value for ϵ. We immediately find the solutions, which are

$$\begin{aligned}(x_0, y_0) &= (0, 0),\\ (x_1, y_1) &= \left(-\frac{b\epsilon}{2} + \frac{\epsilon}{2}\sqrt{b^2 - 4a\epsilon}, 0\right),\\ (x_2, y_2) &= \left(-\frac{b\epsilon}{2} - \frac{\epsilon}{2}\sqrt{b^2 - 4a\epsilon}, 0\right).\end{aligned}$$

The value of parameters such that a degeneracy occurs at those equilibria are as follows,

1. the equilibrium x_0 is degenerate with a zero eigenvalue when $a = 0$,
2. the equilibrium x_1 (and also x_2) is degenerate with a zero eigenvalue when $a = 0$ and $b > 0$ ($a = 0$ and $b < 0$ respectively),
3. the equilibria x_1 and x_2 are degenerate with a zero eigenvalue when $a = \frac{\epsilon b^2}{4}$.

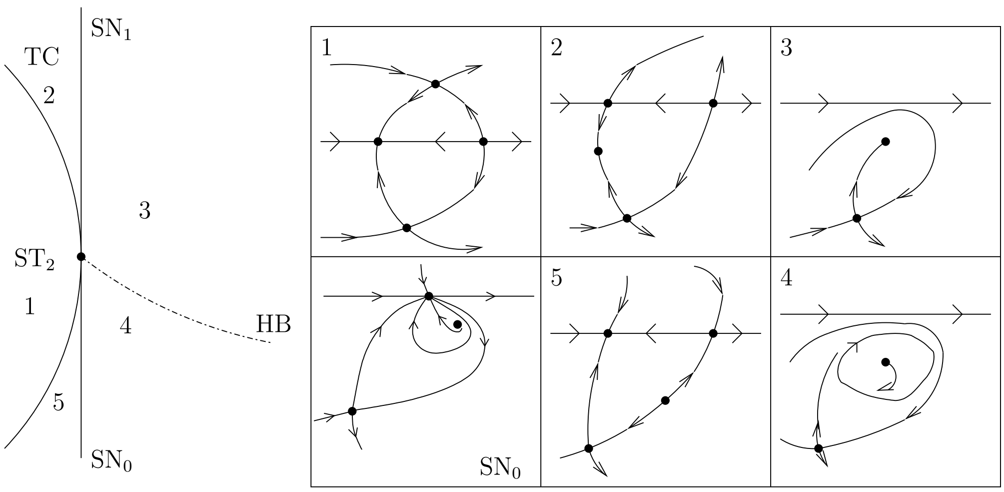

Figure 2.5: The dynamics around the second interaction of the Lotka-Volterra system for the elliptic case. We note here that the periodic orbit, created through Hopf bifurcation and shown in area 4, becomes a homoclinic orbit when we cross to area 5. We have a saddle-node–homoclinic situation. The label SN_1 represents the normal saddle-node bifurcation while the label SN_0 represents a saddle-node homoclinic bifurcation. The other labels are explained in Table 2.1.

The equilibria (x_1, y_1) and (x_2, y_2) coalesce in a saddle-node bifurcation along $a = \frac{\epsilon b^2}{4}$, and either of them crosses the equilibrium (x_0, y_0) in a transcritical bifurcation along $a = 0$. We set $f(x, a) = ax + bx^2 + \epsilon x^3$ to check the non-degeneracy conditions of the saddle-node bifurcation along $a = b^2/4$:

$$\frac{\partial f}{\partial a}(x_1, \epsilon b^2/4) = \frac{\partial f}{\partial a}(x_2, \epsilon b^2/4) = -\frac{\epsilon b}{2} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x_1, \epsilon b^2/4) = \frac{\partial^2 f}{\partial x^2}(x_2, \epsilon b^2/4) = -b, \quad (2.28)$$

and those of the transcritical bifurcation along $a = 0$:

$$\frac{\partial f}{\partial a}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial a \partial x}(0, 0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(0, 0) = 2b. \quad (2.29)$$

Thus, it follows from the computation above, that in the plane of parameters a and b, a codimension-one saddle-node bifurcation takes place along the line $a = \epsilon b^2/4$ and a transcritical bifurcation takes place along the line $a = 0$.

The equilibrium (x_0, y_0) also undergoes a Hopf degeneracy (i.e. the linearized matrix evaluated at (x_0, y_0) has a pair of purely imaginary eigenvalues) for $b = 0$ and $a < 0$. For $b = 0$ and $a > 0$ the equilibrium is a neutral saddle. Along the Hopf degeneracy condition the Lyapunov coefficient is given by $l_1 = 1/(4a\sqrt{|a|})$. Thus, we conclude that a codimension-one Hopf bifurcation occurs along the line $b = 0$ and $a < 0$ in the plane of parameters a and b.

The only point where these bifurcations are degenerate is $a = b = 0$, at which a single equilibrium exists at the origin. At this point (i.e. $a = b = 0$), the minimal model (2.27) becomes the normal form of degenerate Bogdanov-Takens bifurcations for which the topological phase portrait of the degenerate BT bifurcation has been categorized (see Dumortier et al. [31]) as follows,

- for $\epsilon = 1$, the origin is a topological saddle,

- for $\epsilon = -1$, the origin is a topological focus if $k_2^2 + 8\epsilon < 0$,

- for $\epsilon = -1$, the origin is a topological elliptic point if $k_2^2 + 8\epsilon > 0$.

However, in the next section where we show the relation of the minimal model (2.27) and the Lotka-Volterra system with a constant term, only saddle and elliptic cases occur in the Lotka-Volterra system. This is why in Lotka-Volterra systems with a constant term, there are two topologically different bifurcation diagrams. Thus, we are not going to discuss the focus case of the minimal model.

Up to this point, we understand that the minimal model (2.27) undergoes a saddle-node–transcritical double-zero interaction as the saddle-node bifurcation and the transcritical bifurcation coincide and form a double-zero degeneracy. Emanating from the double-zero interaction is the codimension-one Hopf bifurcation that goes to left hand side of the bifurcation diagram. In Figure 2.6, these local bifurcations are depicted. The scenario depicted in Figure (2.6) is actually the same scenario that was presented in the bifurcation diagram of Lotka-Volterra models with a constant term in Figure 2.1, or in Figures 2.4 and 2.5 for a closer view near the double-zero interaction. Both models have a double-zero interaction of saddle-node and transcritical bifurcations and a Hopf bifurcation emanating from this interaction point.

We remark that we only perform partial bifurcation analysis of the minimal model. We only show that the minimal model has the same basic bifurcation diagram (which are saddle-node, transcritical and Hopf bifurcations) as the Lotka-Volterra model does. In the minimal model, a secondary Hopf bifurcation from equilibria x_1 and x_2, a bifurcation of periodic orbits and a global bifurcations such as a homoclinic connection and a heteroclinic link, might also take place in our bifurcation diagram, depending on the choices of k_1 and k_2.

2.4.2 Relation to the degenerate Bogdanov-Takens normal form

We have discussed that at the point $a = b = 0$, the system (2.27) becomes the normal form of the degenerate Bogdanov-Takens bifurcation. However, the unfolding of the minimal model

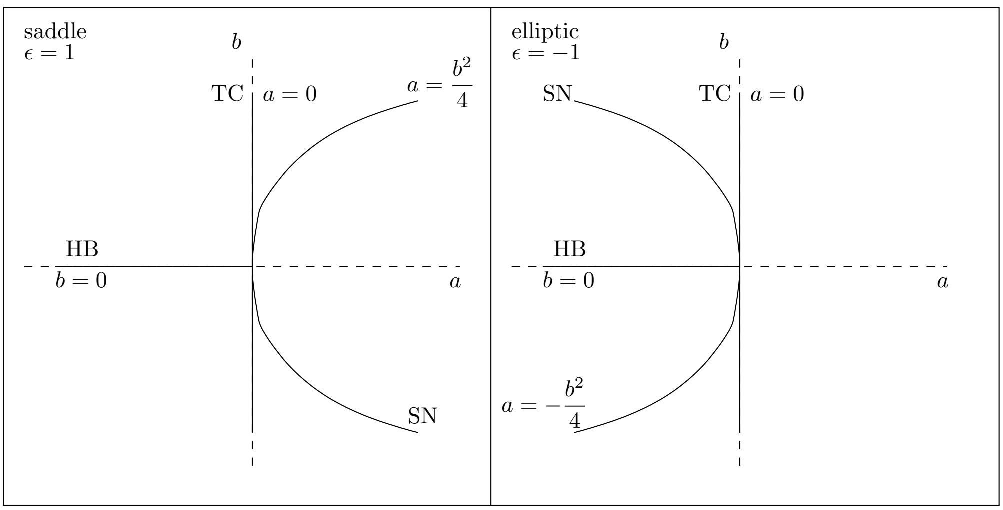

Figure 2.6: The basic bifurcation set for the minimal model (2.27). Labels are explained in Table 2.1.

(2.27) can also be transformed into the unfolding of the codimension-three Bogdanov-Takens bifurcation. The transformation

$$\begin{aligned} z_1 &= x + \frac{\epsilon}{3}b - \frac{2\epsilon}{3k_2}xb - \frac{\epsilon}{27k_1}b^2, \\ z_2 &= y - \frac{2\epsilon}{3k_2}by, \\ \bar{a} &= a - \frac{\epsilon}{3}b^2, \\ \bar{b} &= b - \frac{1}{9k_1}b^2, \end{aligned} \tag{2.30}$$

brings the minimal model in the form of the standard unfolding of the degenerate Bogdanov-Takens bifurcation, truncated up to terms of order three:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= \mu_1 + \mu_2 z_1 + \nu z_2 + k_2 z_1 z_2 + z_1^2 z_2 + \epsilon z_1^3, \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} \mu_1 &= -\frac{\epsilon}{3}\left(\bar{a} + \frac{\epsilon}{9}\bar{b}^2\right)\bar{b}, \\ \mu_2 &= \bar{a}, \\ \nu &= \left(k_1 - \frac{\epsilon}{3}k_2\right)\bar{b}. \end{aligned} \tag{2.32}$$

The local partial bifurcation diagram of the normal form of the degenerate Bogdanov-Takens bifurcation (2.31) is schematically depicted in Figure 2.7. The codimension-one saddle-node bifurcation (labelled SN) and the codimension-one Hopf bifurcation (labelled HB) are now surfaces in the three-dimensional space of parameters μ_1, μ_2 and ν. Equations, used to draw saddle-node and Hopf bifurcation surfaces are:

$$27\mu_1^2 + 4\epsilon\mu_2^3 = 0,$$

and

$$\mu_1^2 + 3\epsilon k_2\mu_1\nu - k_2\mu_1\mu_2 + \epsilon k_2^2\mu_2\nu - \epsilon k_2^3\mu_1 + \mu_2^2\nu - 2\epsilon\mu_2\nu^2 + \nu^3 = 0$$

respectively. The former equation is derived from solving the equation to find equilibria of the system (2.31) and the condition such that the Jacobian matrix evaluated as those equilibria has a single zero eigenvalue. While, the latter equation is derived from solving the equation to find equilibria of the system (2.31) and the condition such that the trace of the Jacobian matrix evaluated as those equilibria is zero.

The codimension-two Bogdanov-Takens bifurcation (labelled BT) is now a curve that is formed from the interaction of the saddle-node and the Hopf bifurcation surfaces. The degenerate Bogdanov-Takens bifurcation (labelled BT_3) which is codimension-three, is the origin of this parameter space. We note that this bifurcation set is partial. We do not draw the complete bifurcation diagram since our main interest is to show the relation between the minimal model and the normal form of codimension-three BT, especially the non-versal unfolding of saddle-node–transcritical interaction.

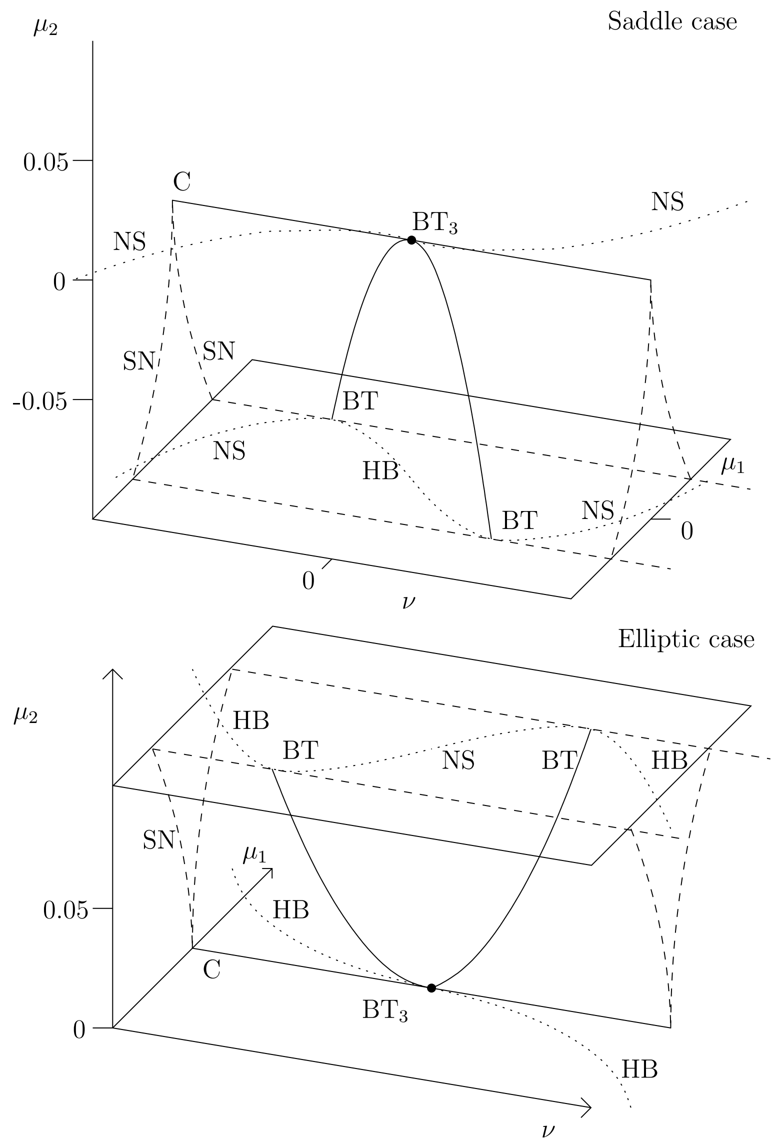

Figure 2.7: The schematic partial bifurcation diagrams of (2.31) for $\epsilon = 1$ (top) and $\epsilon = -1$ (bottom). BT$_3$ is a codimension-three Bogdanov-Takens bifurcation and NS represents a neutral saddle condition. The rest of the labels are the same as listed in Table 2.1.

We note that in the bifurcation diagram in Figure 2.7, the label NS denotes a condition for neutral saddle which is not a bifurcation. The NS condition shares surfaces with a Hopf bifurcation. In Figure 2.7 (top) we see that the Hopf bifurcation is in the interior of the surface while in Figure 2.7 (bottom) the neutral saddle condition is the one that is in the interior.

Now we can start to analyse the double-zero interaction of the minimal model, considered as a non-versal unfolding of the degenerate BT normal form. The parameter space a and b is mapped by the transformation (2.32) to the parameter space μ_1, μ_2 and ν, which is three-dimensional. Thus, we will have a surface in three-dimensional space, parameterized by a and b. Instead of embedding the parameterized surface in the three-dimensional bifurcation diagram, we take a two-parameter slice of the bifurcation diagram in Figure 2.7.

Figure 2.8 (left) presents a two-parameter slice of the bifurcation diagram in Figure 2.7 for $\epsilon = 1$. The values of parameters we choose are $\mu_2 = -0.01$, $k_1 = -1.58113883$

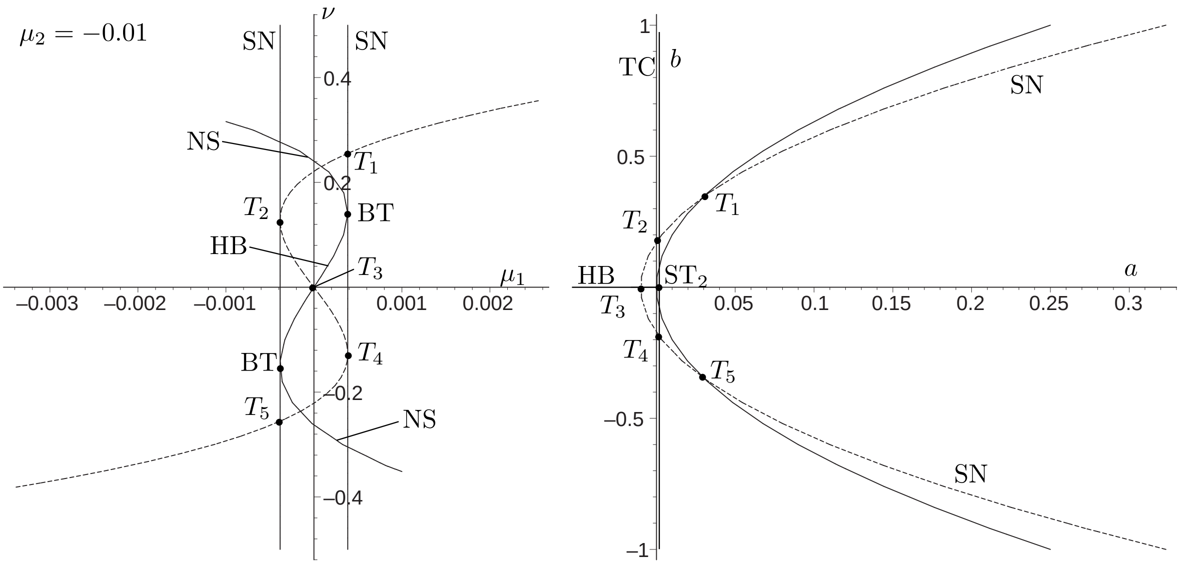

Figure 2.8: Illustration of the saddle-node–transcritical interaction in the saddle case as a non-versal unfolding of the degenerate Bogdanov-Takens bifurcation. Labels are explained in Table 2.1. NS represents a neutral saddle condition. (Left) A two parameter slice of the three-dimensional bifurcation diagram of Figure 2.7. The dotted curve denotes the non-versal unfolding of parameters in the minimal model. (Right) The bifurcation diagram of the minimal model for $\epsilon = 1$. The dotted curve denotes the preimage of the other dotted curve depicted in the left figure under the transformation (2.32).

and $k_2 = -2.529822128$. The values of k_1 and k_2 that are chosen depend on the values of parameters in the Lotka-Volterra system with a constant term (2.1). We will explain the relation between the Lotka-Volterra model and the minimal model in the next section. The vertical solid lines are the saddle-node bifurcation while the solid curve represents the Hopf bifurcation. The dotted curve comes from the intersection of the plane $\mu_2 = -0.01$ and the parameterized surface of the map (2.32). The dotted curve intersects the saddle-node bifurcation four times; two times transversally (denoted by T_1 and T_5) and two times in a tangency (denoted by T_2 and T_4). It also intersects the Hopf bifurcation curve transversally which is denoted by T_3. The preimage of this curve under the transformation (2.32) that is in the parameter space a and b is depicted in Figure 2.8 (right). The preimage curve also intersects the saddle-node bifurcation twice (T_1 and T_5), the transcritical bifurcation twice (T_2 and T_4) and the Hopf bifurcation in T_3.

Figure 2.9 (left) presents another two-parameter slice of the three dimensional bifurcation diagram in Figure 2.7. The values of parameters we choose are $\epsilon = -1$, $\mu_2 = 0.01$, $k_1 = -1.870828693$ and $k_2 = 4.276179871$. Note that we choose μ_2 positive since the surface of the saddle-node bifurcation is upside down in the elliptic case. The dotted curve in Figure 2.9 (left) is the intersection of the plane $\mu_2 = 0.01$ and the parameterized surface of the map (2.32). As in the saddle case, this curve also intersects the saddle-node bifurcation four times, two times transversally (denoted by T_1 and T_4) and two times in a tangency (denoted by T_2 and T_3). However, it does not intersect the Hopf bifurcation. This fact agrees with the

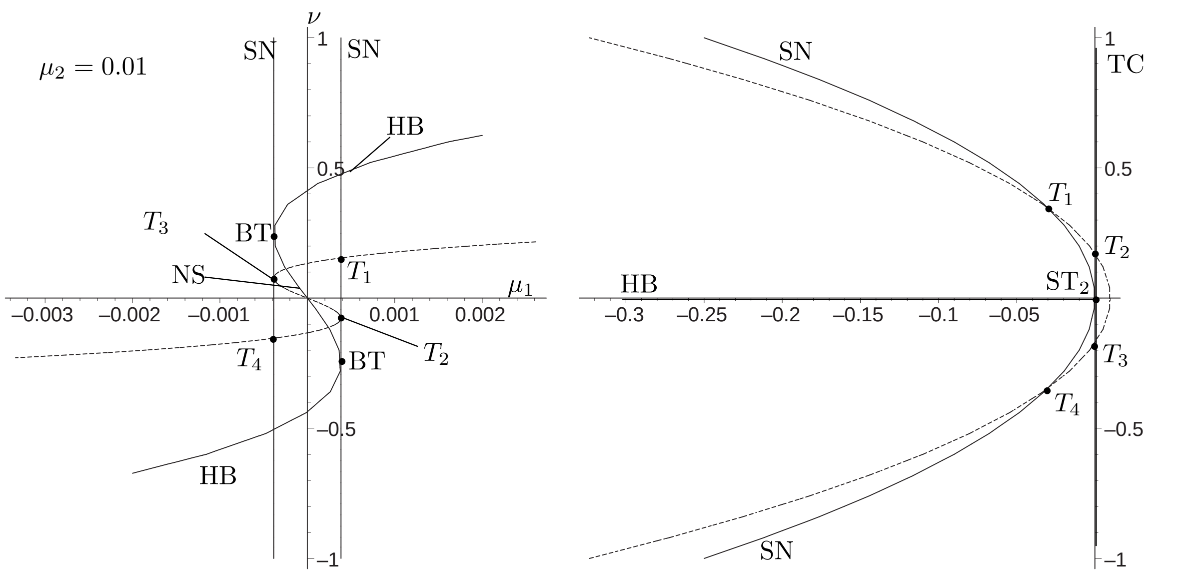

Figure 2.9: Illustration of the saddle-node–transcritical interaction in the elliptic case as a non-versal unfolding of the degenerate Bogdanov-Takens bifurcation. Labels are explained in Table 2.1. NS represents a neutral saddle condition. (Left) A two parameter slice of the three-dimensional bifurcation diagram of Figure 2.7. The dotted curve denotes the non-versal unfolding of parameters in the minimal model. (Right) The bifurcation diagram of the minimal model for $\epsilon = -1$. The dotted curve denotes the preimage of the other dotted curve in the left figure under the transformation (2.32).

preimage of this curve under the transformation (2.32) in the parameter space a and b as we can see in Figure 2.9 (right).

2.4.3 Equivalence to the Lotka-Volterra model with a constant term

The saddle-node–transcritical bifurcation with double-zero eigenvalues occurs in the Lotka-Volterra model with a constant term when

$$\begin{aligned} x_1 &= x_1^* = -\frac{b_1}{2a_{11}}, & e &= e^* = \frac{b_1^2}{4a_{11}}, \\ x_2 &= x_2^* = 0, & b_2 &= b_2^* = \frac{b_1 a_{21}}{2a_{11}}. \end{aligned} \tag{2.33}$$

We introduce $u_1 = x_1 - x_1^*$, $u_2 = x_2 - x_2^*$, $p_1 = e - e^*$ and $p_2 = b_2 - b_2^*$, thus we have

$$\begin{aligned} \dot{u}_1 &= f_1(u_1, u_1, p_1, p_2) = \gamma u_2 + a_{11}u_1^2 + a_{12}u_1u_2 + p_1, \\ \dot{u}_2 &= f_2(u_1, u_1, p_1, p_2) = a_{21}u_1u_2 + a_{22}u_2^2 + p_2u_2, \end{aligned} \tag{2.34}$$

where $\gamma = -b_1a_{12}/(2a_{11})$. Now consider a transformation given by

$$\begin{aligned} v_1 &= u_1 - \frac{a_{22}}{a_{21}\gamma}p_1 + \frac{1}{a_{21}}p_2 + \phi_1(u_1, u_2, p_1, p_2), & q_1 &= p_1 - \frac{2\gamma a_{11}}{D_3}p_2 + \psi_1(p_1, p_2), \\ v_2 &= \gamma u_2 + p_1 + \phi_2(u_1, u_2, p_1, p_2), & q_2 &= p_1 + \psi_2(p_1, p_2), \end{aligned} \tag{2.35}$$

where the smooth functions $\phi_{1,2}$ and $\psi_{1,2}$ are polynomial functions in terms of the coordinates and the parameters. Clearly, this transformation is smooth and invertible on an open neighbourhood of the codimension-two point. We then choose

$$
\begin{aligned}
\phi_1(u_1,u_2,p_1,p_2) = {}& \frac{a_{21}d_2^2 - a_{22}D_2}{\gamma^2 a_{21} D_4} p_1 u_1 + \frac{D_3}{\gamma a_{21} D_4} p_2 u_1 + \frac{a_{22} D_3 (D_3 - a_{12}a_{21})}{2\gamma^2 a_{21}^3 D_4} p_1 u_2 \\
&+ \frac{d_2 D_3}{\gamma a_{11} a_{21} D_4} p_2 u_2 - \frac{d_2}{2\gamma} u_1^2 - \frac{d_1 D_3}{2\gamma a_{21}^3 D_4} p_2^2 + \frac{D_3(2a_{11}D_2 - a_{21}D_3)}{2\gamma^2 a_{21}^3 a_{11} D_4} p_1 p_2 \\
&+ \frac{D_3^2 D_2 - d_2 a_{21}^2 (3d_2 a_{12} a_{21} - 2a_{22} D_3 + 2D_1 a_{12})}{4\gamma^3 a_{21}^3 D_4} p_1 u_1^2 + \frac{d_2 d_3}{6\gamma^2} u_1^3, \\
\psi_1(p_1,p_2) = {}& \frac{a_{22} D_1}{\gamma^2 a_{21}^2} p_1^2 - \frac{D_2}{\gamma a_{21}^2} p_1 p_2 + \frac{a_{11}}{a_{21}^2} p_2^2, \\
\psi_2(p_1,p_2) = {}& -\frac{D_1 a_{22}}{a_{21}^2 \gamma^2} p_1^2 + \frac{a_{11}}{a_{21}^2} p_2^2,
\end{aligned}
\tag{2.36}
$$

where $D_4 = 2a_{11} + a_{21}$, $d_1 = 2a_{11} - a_{21}$, $d_2 = a_{22} + a_{12}$ and $d_3 = 2a_{12} + a_{22}$. Finally, we choose

$$\phi_2(u_1,u_2,p_1,p_2) = a_{11}u_1^2 + a_{12}u_1u_2 + f_1\partial_{u_1}\phi_1 + f_2\partial_{u_2}\phi_1.$$

Up to the third order this yields

$$\dot{v}_1 = v_2, \tag{2.37}$$

$$\dot{v}_2 = \frac{D_3}{\gamma a_{21}} q_1 v_2 - a_{21} q_2 v_1 + D_4 v_1 v_2 - \frac{D_3}{\gamma} q_1 v_1^2 + \frac{D_3}{2\gamma} q_2 v_1^2 - \frac{D_3}{2\gamma} v_1^2 v_2 - a_{11} a_{21} v_1^3. \tag{2.38}$$

In the second step we apply a near-identity transformation allowing for a smooth reparametrization of time:

$$
\begin{aligned}
w_1 &= v_1 + \frac{D_3^2}{3\gamma^2 a_{21}^2 D_4} q_1 v_1 - \frac{D_3}{6\gamma a_{21}} v_1^2 - \frac{D_3^3}{18\gamma^3 a_{21}^3 D_4} q_1 v_1^2, \\
w_2 &= v_2 + \frac{D_3^2}{3\gamma^2 a_{21}^2 D_4} q_1 v_2,
\end{aligned}
\tag{2.39}
$$

$$t' = t\left(1 - \frac{D_3}{3\gamma a_{21}} v_1,\right) \tag{2.40}$$

which results in

$$
\begin{aligned}
\dot{w}_1 &= w_2, \\
\dot{w}_2 &= \frac{D_3}{\gamma a_{21}} q_1 w_2 - a_{21} q_2 w_1 + D_4 w_1 w_2 - \frac{D_3}{\gamma} q_1 w_1^2 + \frac{a_{11} D_3}{\gamma a_{21}} w_1^2 w_2 - a_{11} a_{21} w_1^3.
\end{aligned}
\tag{2.41}
$$

Finally, we rescale the variables as

$$
\begin{aligned}
\bar{w}_1 &= \operatorname{sign}(a_{11}a_{21}) \frac{D_3}{a_{21}^2 \gamma} \sqrt{|a_{11}a_{21}|}\, w_1, & \bar{q}_1 &= -\operatorname{sign}(a_{11}a_{21}) \frac{D_3^2}{\gamma^2 a_{21}^2 \sqrt{|a_{11}a_{21}|}}\, q_1, \\
\bar{w}_2 &= \frac{D_3^2}{a_{21}^4 \gamma^2} \sqrt{|a_{11}a_{21}|}\, w_2, & \bar{q}_2 &= -\frac{D_3^2}{\gamma^2 a_{21}^3}\, q_2, \\
\bar{t} &= \operatorname{sign}(a_{11}a_{21}) \frac{a_{21}^2 \gamma}{D_3}\, t,
\end{aligned}
\tag{2.42}
$$

and we find model (2.27) with

$$
\begin{aligned}
x &= \bar{w}_1, & y &= \bar{w}_2, \\
a &= \bar{q}_2, & b &= \bar{q}_1, \\
\epsilon &= -\operatorname{sign}(a_{11}a_{21}), & k_1 &= -\frac{\sqrt{|a_{11}a_{21}|}}{a_{21}}, \\
k_2 &= \frac{D_4}{\sqrt{|a_{11}a_{21}|}}.
\end{aligned}
\tag{2.43}
$$

An immediate consequence of the transformation above is the fact that the focus case does not appear in the Lotka-Volterra system with a constant term. From the classification of the degenerate BT bifurcation, the focus case occurs when $\epsilon = -1$ and $k_2^2 + 8\epsilon < 0$. Thus we shall show that when $\epsilon = -1$, the term "$k_2^2 + 8\epsilon$" is always greater than zero. Firstly we substitute the values of ϵ and k_2 in terms of the parameters in Lotka-Volterra models using the equation (2.43). The fact that $\epsilon = -1$ implies $a_{11}a_{21} > 0$, gives

$$\begin{aligned} k_2^2 + 8\epsilon &= (\frac{D_4}{\sqrt{|a_{11}a_{21}|}})^2 + 8(-1), \\ &= (\frac{2a_{11} + a_{21}}{\sqrt{|a_{11}a_{21}|}})^2 - 8, \\ &= (4a_{11}^2 - 4a_{11}a_{21} + a_{21}^2)/(a_{11}a_{21}), \\ &= (2a_{11} - a_{21})^2/(a_{11}a_{21}), \\ &\geq 0. \end{aligned} \tag{2.44}$$

Therefore, we conclude that the focus case never occurs in the Lotka-Volterra system with a constant term.

2.4.4 The minimal model with an invariant manifold

When we compare the bifurcation diagrams of the Lotka-Volterra model to those of the minimal model, we see that the local bifurcations are topologically equivalent. The global bifurcations, however, are not. Most notably, the diagram of the Lotka-Volterra model in the saddle case has a heteroclinic loop bifurcation in which a periodic orbit is created, whereas the corresponding diagram of the minimal model has two separate simple heteroclinic bifurcations that do not involve a periodic orbit. Instead, the periodic orbit is created in a homoclinic bifurcation. The reason is that the Lotka-Volterra model has a special structure. The x-axis is invariant for all parameter values. If two saddle points exist on this axis then part of the axis forms a structurally stable heteroclinic connection. This structure is not conserved by the transformation to the third order minimal model. In this section we will show that the complete bifurcation structure of the Lotka-Volterra model can be preserved by adding a fourth order term to the minimal model.

In order to correctly model both the local and the global bifurcations of the Lotka-Volterra model with a constant term, we need to add a fourth order term to the minimal model:

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = ax + k_1by + bx^2 + k_2xy + x^2y + \epsilon x^3 + k_3x^4, \end{aligned} \tag{2.45}$$

This extended model admits an invariant manifold given by

$$y = g(x) = ak_1 + bk_1x + \epsilon k_1x^2 + \frac{1}{3}x^3$$

under the conditions

$$2\epsilon k_1^2 - k_1k_2 - 1 = 0 \tag{2.46}$$

$$k_1k_3 - \frac{1}{3} = 0 \tag{2.47}$$

The first condition is indeed satisfied by relations (2.43). If we extend the computation of the transformation of the Lotka-Volterra model to the minimal model to fourth order we also find that the second condition is satisfied. Also, there are three of the equilibrium points of the extended minimal model that lie on the invariant manifold. Thus, the special structure of the Lotka-Volterra model is preserved.

The resulting bifurcation diagrams are shown below for the saddle and the elliptic case (see Figures 2.10 and 2.11). We used the same values of k_1 and k_2 that have been computed before. We have already shown that k_1 and k_2 satisfy condition (2.46). The value of k_3 is chosen such that the condition (2.47) is satisfied. In the former case the heteroclinic loop now appears as in the Lotka-Volterra model (see Figure 2.4). In the latter case, the periodic orbit that is born through a Hopf bifurcation in area 2, is also terminated when we cross to area 3 through a saddle-node bifurcation. The saddle-node homoclinic orbit is preserved in the fourth order minimal model.

However, the transformation (2.35) that we have performed in the previous section only brings the Lotka-Volterra model to the first minimal model (2.27) which is correct up to order three. This is mainly because the transformation (2.35) is up to order three as well. We hope in future to extend this transformation up to order four such that we can bring the Lotka-Volterra model to the second minimal model (2.45), as this model represents the complete bifurcation of the Lotka-Volterra model and this model also has a relation to the degenerate codimension-three BT bifurcation.

2.5 Discussion

The Lotka-Volterra system with a constant term (2.1) indeed has interesting bifurcations as shown in the bifurcation diagrams in Figure 2.1. Our main interest in this chapter has been to explain bifurcations that occur in this system, in particular the interactions of saddle-node

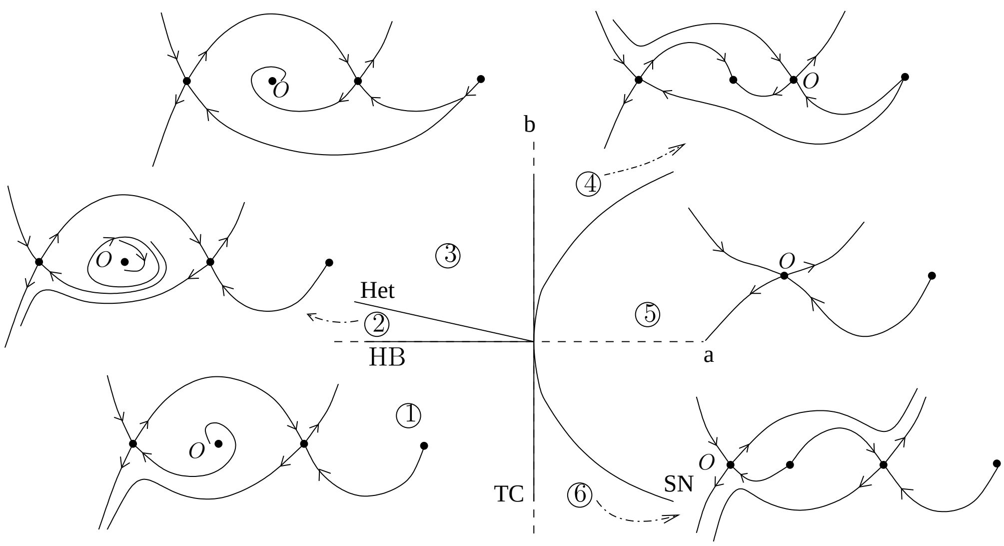

Figure 2.10: Bifurcation diagram of the saddle-case of the system (2.45).

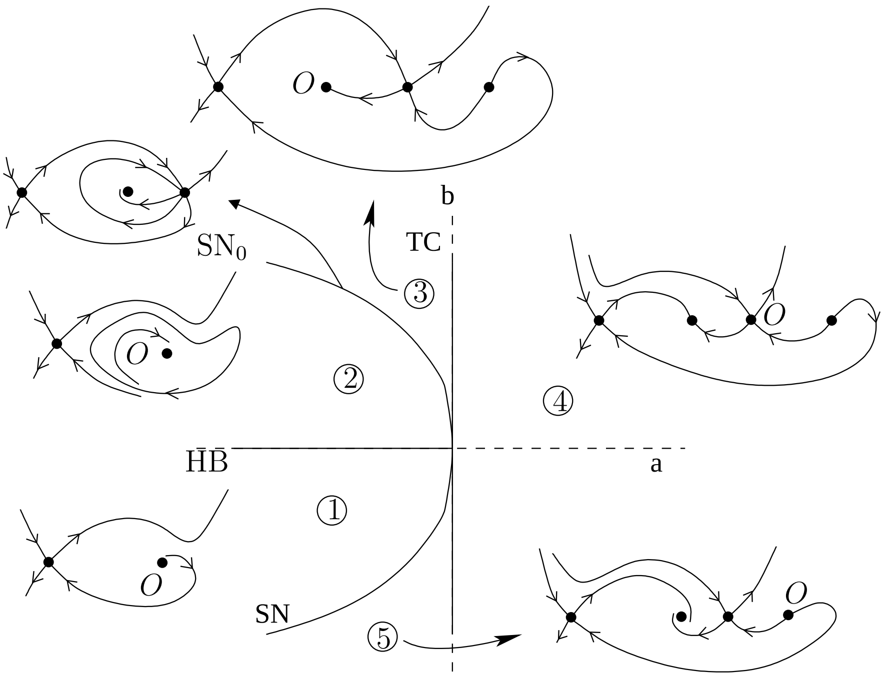

Figure 2.11: Bifurcation diagram of the elliptic-case of the system (2.45).

and transcritical bifurcations. There are two interactions, one of which has a single-zero degeneracy and the other has a double-zero degeneracy. However, these degeneracies do not produce the common bifurcations that a vector field usually undergoes when it has the same degeneracies. Thus, all the analyses we have done in this chapter try to explain the unusual bifurcations that occur in (2.1).

The first interaction turns out to have the same degeneracies as a cusp bifurcation. In this chapter we have shown that the unfolding of the first interaction can be represented by a minimal model (2.10). We have also found transformations that bring the first interaction to the minimal model and bring the minimal model to the unfolding of a cusp bifurcation respectively. Thus, the first interaction of saddle-node and transcritical bifurcations is a non-versal unfolding of a cusp bifurcation. The normal form of a cusp bifurcation is unfolded with the two parameters and so is the case for the first interaction of saddle-node and transcritical bifurcations.

The second interaction is more involved. It turns out that the second saddle-node–transcritical interaction has the same degeneracies as a codimension-three Bogdanov-Takens bifurcation (DBT). We have introduced a minimal model (2.27) that undergoes the same set of bifurcations as the second interaction in the Lotka-Volterra system does under some conditions on the parameters. We have shown in this chapter that the second saddle-node–transcritical interaction is a non-versal unfolding of degenerate BT. However, we still use the term "non-versal" even though the number of unfolding parameters in the second interaction is not the same as the unfolding of DBT.

In this chapter we have shown that common bifurcations can be unfolded in a different way producing different bifurcations, just as Lotka-Volterra systems have done. Hence, when one does a bifurcation analysis of some vector field and finds unusual bifurcations, it can be explained by using the same method that we have just used in this chapter. One can explain that the unfolding of the vector field studied is non-versal such that it has a set of unusual bifurcations. However, this analysis triggers another question which is: why we have such bifurcations in the first place. It turns out that when we have such unusual bifurcations, the system must possess a special structure. The Lotka-Volterra system that has undergone such unusual bifurcations must also have a special structure. This topic will be discussed in the next chapter.

CHAPTER 3

Bifurcation analysis of systems having a codimension-one invariant manifold

3.1 Introduction

As noticed in Wiggins [126, chapter 13], a *special structure* in a dynamical system greatly constrains the type of dynamics that are allowed, and it also provides techniques of analysis that are particular to dynamical systems with the special structure. These particular special structures are also important because they arise in a variety of applications. One important example of a dynamical system possessing a special structure is the Hamiltonian vector field. Over the past years there has been a great deal of research on Hamiltonian systems. Most research has occurred along two directions. One direction is concerned with the geometrical structure of Hamilton's equations. The other direction is concerned with the dynamical properties of the flow generated by Hamiltonian vector fields. An excellent book on classical mechanics, for example Abraham and Marsden [1], will outline the background for both view points. Another example of a system with a special structure is a vector field possessing a symmetry. This is also a broad research area in dynamical systems theory. Symmetry plays an essential role in studying the theory and applications of dynamical systems, in particular the influence of symmetry on normal forms, bifurcation diagrams, amongst others, see Vanderbauwhede [121]. The symmetry property can also be used to help reduce problems in another system with a special structure as we can see in Tuwankotta and Verhulst [119]. One type of symmetry that often arises in applications is the reversing symmetry (see Roberts and Quispel [105] and Lamb [82]). The reversing symmetry is also studied in bifurcation theories [84], and in physics [83]. In this chapter we consider one type of dynamical system which is an ordinary differential equation having a special structure, namely a codimension-one invariant manifold.

3.1.1 Setting up the problem

Let us first define our problem. Suppose that we have smooth dynamical systems and we would like to perform some (mainly bifurcation theory) analysis of the dynamics. Consider an n-dimensional vector field, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} \in \mathcal{C}^k(\mathbb{R}^n)$ for some k, having a codimension-one invariant manifold. A codimension-one invariant manifold is an invariant submanifold M of dimension $n-1$ inside an n-dimensional manifold. We are interested in a local bifurcation analysis, say near an equilibrium. Let us assume that the equilibrium is the origin $\mathbf{0}$. Without loss of generality, near $\mathbf{0}$, the ambient manifold is $\mathbb{R}^n$ and the invariant manifold is $\mathbb{R}^{n-1}$, given in coordinates of the original manifold by saying $M = \{(x_1, x_2, \ldots, x_{n-1}, y) | y = 0\}$ if $(x_1, x_2, \ldots, x_{n-1}, y)$ are the $\mathbb{R}^n$ coordinates. And we assume that any smooth codimension-one manifold can be rectified this way.

Example 3.4 (One-dimensional case)**.** *Suppose we have a one-dimensional dynamical system having a codimension-one invariant manifold. The origin,* $\mathbf{0}$ *will be our invariant manifold. Locally, we can define our dynamical system as follows,*

$$\dot{y} = y f(y) \quad y \in \mathbb{R}, \tag{3.1}$$

where $f(y)$ *is some function in* $\mathbb{R}$.

Example 3.5 (n-dimensional case)**.** *Suppose that we have an* n*-dimensional vector field,* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ *having a codimension-one invariant manifold, then the vector field can be written as follows,*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_{n-1}, y), \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_{n-1}, y), \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}, y), \\ \dot{y} &= y f_n(x_1, x_2, \ldots, x_{n-1}, y). \end{aligned} \tag{3.2}$$

The invariant manifold M *is invariant with respect to the above differential equation, (i.e.* M *is said to be invariant under the vector field* $\dot{\xi} = \phi(\xi)$ *if for any* $\xi_0 \in M \subset \mathbb{R}^n$ *we have* $\xi(t, \xi_0) \in M$ *for all* $t \in \mathbb{R}$*).*

Let us look at the $n \times n$ Jacobian matrix of the system (3.2), evaluated at the origin $\mathbf{0}$,

$$J(\mathbf{0}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{0}) & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(\mathbf{0}) & \frac{\partial f_1}{\partial y}(\mathbf{0}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{0}) & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{n-1}}(\mathbf{0}) & \frac{\partial f_2}{\partial y}(\mathbf{0}) \\ \vdots & & \ddots & & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1}(\mathbf{0}) & \frac{\partial f_{n-1}}{\partial x_2} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}}(\mathbf{0}) & \frac{\partial f_{n-1}}{\partial y}(\mathbf{0}) \\ 0 & 0 & \cdots & 0 & f_n(\mathbf{0}) \end{pmatrix} \tag{3.3}$$

The eigenvalues of the above matrix are $f_n(\mathbf{0})$ and all eigenvalues of the $(n-1) \times (n-1)$ matrix below,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{0}) & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n-1}}(\mathbf{0}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{0}) & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{n-1}}(\mathbf{0}) \\ \vdots & & & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1}(\mathbf{0}) & \frac{\partial f_{n-1}}{\partial x_2} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}}(\mathbf{0}) \end{pmatrix}. \tag{3.4}$$

The simplest degeneracies are a single-zero eigenvalue and a pair of purely imaginary eigenvalue degeneracies. We have our first proposition.

Proposition 3.6. *The center manifold of an equilibrium having a single-zero eigenvalue or a pair of purely imaginary eigenvalue of the matrix (3.4) lies inside the codimension-one invariant manifold up to any desired degree of accuracy.*

Proof. See Appendix A. □

Our general purpose is to describe all the low codimension bifurcations of equilibria and perhaps periodic orbits in such a way that the property of having an invariant manifold M is always preserved and the manifold remains the same. Our bifurcation analysis of dynamical systems with a special structure will be analogous to that of general dynamical systems in which we do not have a special structure. First we want to classify possible low codimension bifurcations. Note that we only discuss codimension-one and codimension-two bifurcations. Codimension-one bifurcations have two types of degeneracy, namely a single zero eigenvalue and a pair of purely imaginary eigenvalues. In general systems, these conditions yield saddle-node and Hopf bifurcations respectively. However, when the system has a codimension-one invariant manifold we may not have such bifurcations.

We will not discuss the cases where the center manifold lies entirely in the invariant manifold M since they correspond to the usual generic bifurcations. Since the complex pair of eigenvalues can only come from the matrix (3.4) the center manifold of the pair of purely imaginary eigenvalues degeneracy lies inside the invariant manifold M. Thus, we shall not analyse the Hopf bifurcation because it will be similar to the generic case. If the single-zero eigenvalue comes from the matrix (3.4) it will also not be of interest for the same reason. Hence the only degeneracy for a codimension-one bifurcation that we are going to discuss is a single-zero eigenvalue degeneracy where $f_n(\mathbf{0}) = 0$.

Codimension-two bifurcations give us more possibilities. We start with the same degeneracy as the codimension-one bifurcations have, but now we will have an additional degeneracy in the nonlinear terms of the normal form. We shall get bifurcations whose codimension are higher by studying a degeneracy that comes from higher order terms or non-linear terms of the function $f_n(x_1, x_2, \ldots, y)$. We then also consider cases in which the linear part of the vector field is doubly degenerate. The eigenvalues of the matrix (3.4) are now allowed to have degeneracies, whether they are zero or purely imaginary. Combined with the first single-zero degeneracy, the center manifold will not lie entirely inside the invariant manifold. These additional degeneracies also allow the system to have codimension-two (or higher) bifurcations. Firstly we shall discuss the double zero eigenvalue degeneracy. In a general system, this corresponds to the Bogdanov-Takens bifurcation. Secondly we have a single zero eigenvalue with a pair of purely imaginary eigenvalues that corresponds to a saddle-Hopf bifurcation in the general system.

3.2 Local codimension-one bifurcations of equilibria

As we analyse earlier, that a local codimension-one bifurcation of equilibria involves a linear part degeneracy that is only a single zero eigenvalue since a pair of purely imaginary eigenvalues is not the case of interest.

Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is an n-dimensional vector field with a codimension-one invariant manifold as expressed in the example 3.5. Suppose that we have a single zero degeneracy, i.e. the Jacobian matrix $D\mathbf{f}(\mathbf{0})$ has a single-zero eigenvalue degeneracy. We assume that there is no other degeneracy. Recall that the case of interest of a single-zero eigenvalue degeneracy occurs when $f_n(\mathbf{0}) = 0$. Using the Center Manifold Theorem, we reduce the dimension of our system as follows,

$$\dot{y} = y\tilde{f}(y) = f(y) \quad y \in \mathbb{R}, \tag{3.5}$$

where $\tilde{f}(y) = f_n(x_1(y), \ldots, x_{n-1}(y), y)$. The Taylor expansion of the function $\tilde{f}(y)$ is given by,

$$\tilde{f} = a_0 + a_1 y + a_2 y^2 + \mathcal{O}(|y|^3), \tag{3.6}$$

where $a_0 = 0$ due to $\frac{df(y)}{dy}(0) = 0$ and $a_1 \neq 0$ since there is no other degeneracy. Thus, we have a one-dimensional vector field,

$$\dot{y} = f(y) = y(a_1 y + a_2 y^2 + \mathcal{O}(|y|^3). \tag{3.7}$$

The vector field above is already in the normal form. We truncate the terms of order 3 and higher, and rescale the coordinate by the following transformation;

$$y \mapsto \frac{y}{|a_1|}$$

to get:

$$\dot{y} = f(y) = y(sy), \tag{3.8}$$

where $s = \pm 1$, depending on the sign of a_1. The phase portrait of this vector field is easy to determined as this is a one-dimensional vector field. If we take $s = 1$ then the origin is asymptotically stable from the left hand side and unstable from the right hand side. Figure 3.1 shows both cases $s = 1$ and $s = -1$. The next step is to unfold this degeneracy by

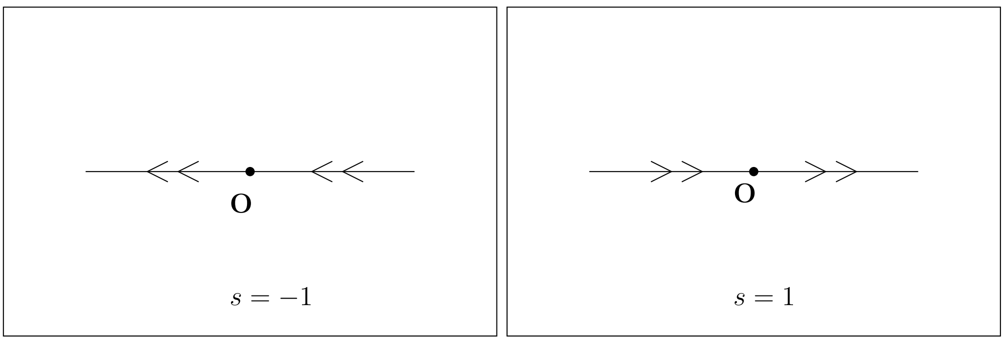

Figure 3.1: The phase portraits of system (3.8) where $s = -1$ (*left*) and $s = 1$ (*right*)

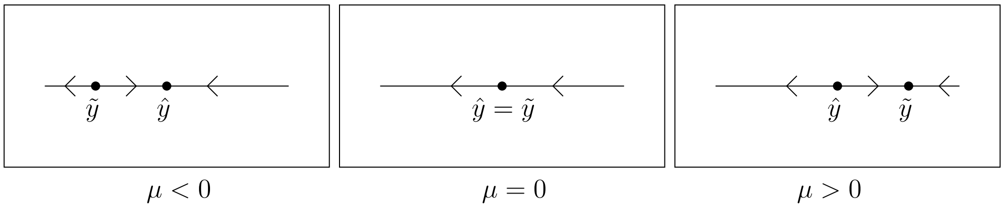

Figure 3.2: Three different phase portraits of (3.9) as we varied μ where $s = -1$

adding parameters in our system. The candidate for our unfolding is a family of vector fields depending on one parameter as follows,

$$\dot{y} = f(y, \mu) = y(\mu + sy), \quad \mu \in \mathbb{R}. \tag{3.9}$$

We only need one parameter to unfold this degeneracy. We can verify that a bifurcation occurs when $\mu = 0$ by checking the single-zero eigenvalue degeneracy of this bifurcation,

$$f(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0. \tag{3.10}$$

Our next task is to do a local and, perhaps a global bifurcation analysis. We choose the case where $s = -1$, while we leave out the other case where $s = 1$, since it can be derived by the same method. We start by computing fixed points of the vector field (3.9),

$$\hat{y} = 0 \quad \text{and} \quad \tilde{y} = \mu. \tag{3.11}$$

We then compute the first derivative to find the linear stability of each fixed point,

$$\left.\frac{\partial f}{\partial y}\right|_{y=\hat{y}} = \mu \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{y=\tilde{y}} = -\mu. \tag{3.12}$$

Hence for $\mu < 0$, the fixed point $y = \hat{y}$ is stable and $y = \tilde{y}$ is unstable. Those two fixed points coalesce at $\mu = 0$ and, for $\mu > 0$, the fixed point $y = \hat{y}$ is unstable and $y = \tilde{y}$ is stable. Thus,

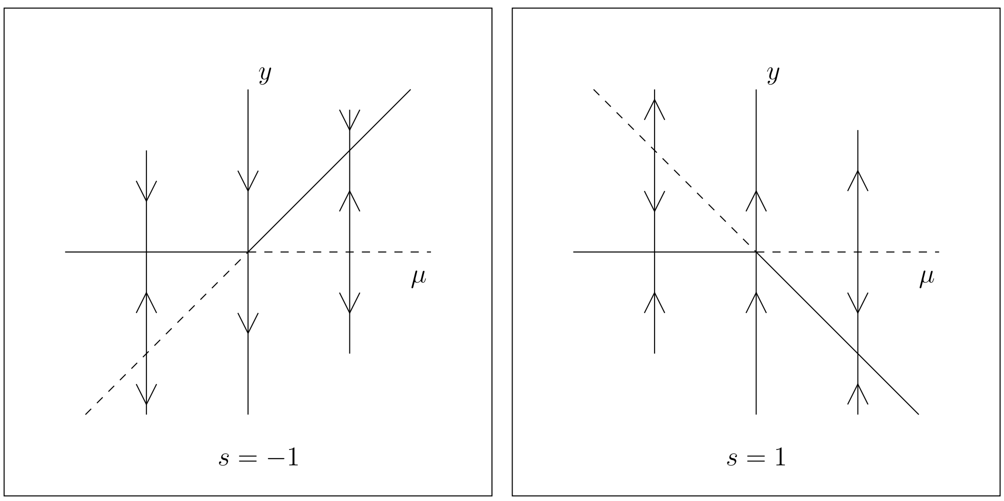

Figure 3.3: One parameter bifurcation diagrams of (3.9), where $s = -1$ (*left*) and $s = 1$ (*right*). The dotted lines and the continuous lines show that the fixed points are unstable and stable, respectively.

an exchange of stabilities has occurred at $\mu = 0$. This type of bifurcation is the so-called the transcritical bifurcation. It is straightforward to check the non-degeneracy conditions of this bifurcation at $\mu = 0$,

$$\frac{\partial f}{\partial \mu}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y \partial \mu}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(0,0) = -2. \tag{3.13}$$

The phase portraits of (3.9) where $s = -1$ as μ varies are depicted in Figure 3.2. The complete bifurcation diagrams of system (3.9) are also depicted in Figure 3.3, in which there are two curves of fixed point that pass through the origin, $(y, \mu) = (0, 0)$. Global dynamics does not occur in this one-dimensional case.

Our final step is now to analyse whether or not higher order terms qualitatively affect the local dynamics near $(y, \mu) = (0, 0)$ of the vector field (3.9). Let us consider a one-parameter family of one-dimensional vector fields,

$$\dot{y} = f(y, \mu) = y\tilde{f}(y, \mu), \tag{3.14}$$

where

$$\tilde{f}(y, \mu) = \mu + a_1 y + \mathcal{O}(|y|^2). \tag{3.15}$$

As the fixed points $y = \hat{y}$ and $y = \tilde{y}$ are hyperbolic, they will persist for small perturbations from higher order terms. Moreover, by the Implicit Function Theorem, the higher order terms do not significantly change the fixed point curves in the bifurcation diagram depicted in Figure 3.3. We conclude that the addition of higher order terms does not introduce any new dynamical phenomena.

We now summarize our result. Let us consider a general one-parameter family of n-dimensional vector fields $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ having the following properties:

1. it has a codimension-one invariant manifold M, preserved under a variation of μ,
2. when $\mu = 0$ the system undergoes only a single-zero degeneracy and
3. the one-dimensional center manifold of this singularity is transversal to the codimension-one invariant manifold.

Then this vector field undergoes a transcritical bifurcation.

3.3 Higher order degeneracy

In the previous section, we considered a local codimension-one bifurcation as a result of a single-zero eigenvalue degeneracy. It turns out that we have a transcritical bifurcation. In this section we discuss a codimension-two bifurcation of an equilibrium that has more than one degeneracy. We consider a single zero degeneracy combined with an additional higher order term degeneracy of the second derivative. We assume that these are the only degeneracies. Using the Center Manifold Theorem, we can reduce the system that has a codimension-one invariant manifold to the one-dimensional center manifold below:

$$\dot{y} = f(y) = y\tilde{f}(y), \quad y \in \mathbb{R}, \tag{3.16}$$

where $f(0) = \frac{df(y)}{dy}(0) = 0$ and $\frac{d^2f}{dy^2}(0) = 0$ due to the singularities we assume. The Taylor expansion of the function $\tilde{f}$ is

$$\tilde{f} = a_0 + a_1 y + a_2 y^2 + \mathcal{O}(|y|^3), \tag{3.17}$$

where $a_0 = a_1 = 0$. Thus we have a normal form of the codimension-two bifurcation of a single-zero eigenvalue with a second order degeneracy:

$$\dot{y} = y(a_2 y^2 + \mathcal{O}(|y|^3)), \tag{3.18}$$

where $a_2 \neq 0$ since there is no other degeneracy. Inside the bracket in the equation above, we truncate terms of order three and higher and rescale the coordinate by the following transformation:

$$y \mapsto \frac{y}{|a_2|}$$

to get:

$$\dot{y} = f(y) = y(sy^2), \tag{3.19}$$

where $s = \pm 1$, depending on the sign of a_2. The dynamics of this vector field are determined by s. We can draw the phase portraits of the above vector field. The system has one fixed point which is asymptotically stable when $s = -1$ and unstable when $s = 1$. We illustrate these phase portraits in Figure 3.4.

We now wish to unfold all the possible behaviour near the fixed point by perturbing this system with parameters provided that we keep preserving the invariant manifold. All these dynamics can be captured by the addition of the lower order term $\mu_1 + \mu_2 y$, so that an unfolding of this degeneracy is represented by

$$\dot{y} = y(\mu_1 + \mu_2 y + sy^2). \tag{3.20}$$

First we compute the fixed points of the system (3.20):

$$y = 0, \quad \mu_1 + \mu_2 y - y^2 = 0, \tag{3.21}$$

where we take the case $s = -1$ and leave out the case $s = 1$ as we have the following symmetry,

$$(y, t, \mu_1, \mu_2, s) \mapsto (-y, -t, -\mu_1, \mu_2, -s). \tag{3.22}$$

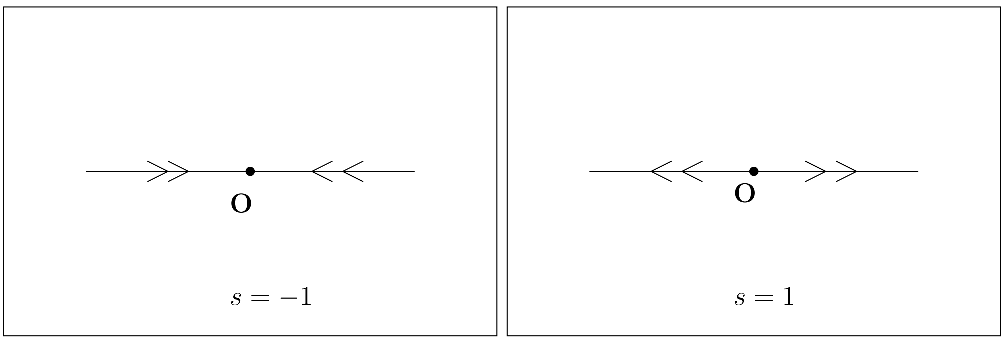

Figure 3.4: The dynamics in the neighbourhood of the origin of the system (3.19) where $s = -1$ (*left*) and $s = 1$ (*right*)

Hence we always have $y_0 = 0$ as our fixed point while the other fixed points can be found by computing:

$$y_{1,2} = \frac{\mu_2 \pm \sqrt{{\mu_2}^2 + 4\mu_1}}{2}. \tag{3.23}$$

From the local analysis above, we conclude that when the term ${\mu_2}^2 + 4\mu_1$ is positive there are three fixed points and when ${\mu_2}^2 + 4\mu_1$ is negative we have the origin as the only fixed point. Furthermore, when the term ${\mu_2}^2 + 4\mu_1$ is zero, the fixed points y_1 and y_2 collide into one equilibrium in a saddle-node bifurcation. We set $f(y, \mu_1) = y(\mu_1 + \mu_2 y - y^2)$. We now check conditions of this bifurcation at $\mu_1 = -{\mu_2}^2/4$,

$$\frac{\partial f}{\partial y}(y_1, -\frac{{\mu_2}^2}{4}) = \frac{\partial f}{\partial y}(y_2, -\frac{{\mu_2}^2}{4}) = 0, \quad \frac{\partial f}{\partial \mu_1}(y_1, -\frac{{\mu_2}^2}{4}) = \frac{\partial f}{\partial \mu_1}(y_2, -\frac{{\mu_2}^2}{4}) = \frac{\mu_2}{2}$$

and

$$\frac{\partial^2 f}{\partial y^2}(y_1, -\frac{{\mu_2}^2}{4}) = \frac{\partial^2 f}{\partial y^2}(y_2, -\frac{{\mu_2}^2}{4}) = -\mu_2. \tag{3.24}$$

Hence, we conclude that a non-degenerate saddle-node bifurcation occurs along the curve ${\mu_2}^2 + 4\mu_1 = 0$ but $\mu_2 \neq 0$.

When $\mu_1 = 0$ and $\mu_2 < 0$ ($\mu_2 > 0$ respectively), the equilibrium y_1 (y_2 respectively) coincides with y_0. The stabilities of these fixed points are determined by:

$$\frac{\partial f}{\partial y}(y, \mu_1) = \mu_1 + 2\mu_2 y - 3y^2. \tag{3.25}$$

Then we compute the eigenvalues of both equilibria,

$$\frac{\partial f}{\partial y}(y_0, \mu_1) = \mu_1 \quad \text{and} \quad \frac{\partial f}{\partial y}(y_{1,2}, \mu_1) = \frac{1}{2}(\mu_2 \pm \sqrt{{\mu_2}^2 + 4\mu_1})\sqrt{{\mu_2}^2 + 4\mu_1}. \tag{3.26}$$

We consider case when $\mu_2 > 0$. When $\mu_1 > 0$ the equilibrium $y = y_0$ is unstable as its eigenvalue is positive and the equilibrium $y = y_2$ is stable as its eigenvalue is negative. On the other hand, when $\mu_1 < 0$ the equilibria $y = 0$ and $y = y_1$ are stable and unstable respectively. Hence, an exchange of stabilities occurs as they coincide when $\mu_1 = 0$ in a transcritical bifurcation. We check the non-degeneracy conditions of this bifurcation,

$$\frac{\partial f}{\partial \mu_1}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y \partial \mu_1}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 2\mu_2, \tag{3.27}$$

to conclude that a non-degenerate transcritical bifurcation occurs along the curve $\mu_1 = 0$ but $\mu_2 \neq 0$.

Thus, we shall have two bifurcation curves in our parameter space, which are the saddle-node and transcritical bifurcations. Both bifurcation curves coincide when $\mu_1 = \mu_2 = 0$ at which the degeneracy of a single-zero eigenvalue with a second order degeneracy occurs. All these dynamics are illustrated in Figure 3.5. We have an interaction of the saddle-node and the transcritical bifurcations at $(\mu_1, \mu_2) = (0, 0)$.

We now analyse the effect of higher order terms. First we put them back in (3.20),

$$\dot{y} = y(\mu_1 + \mu_2 y - y^2 + \mathcal{O}(|y|^3)), \tag{3.28}$$

where $y, \mu_1, \mu_2 \in \mathbb{R}$. The addition of higher order terms does not affect the existence and the stability of the fixed points ($y = y_0$, $y = y_1$ and $y = y_2$) because of the fact that they are hyperbolic. We do not consider other fixed points as they are far away from the origin. Then, by using the Implicit Function Theorem for a sufficiently small neighbourhood of $(\mu_1, \mu_2) = (0, 0)$, higher order terms do not affect the local bifurcation curves which are transcritical and saddle-node bifurcations.

We summarize our result. Let us consider a general two-parameter family of n-dimensional vector fields, i.e. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu_1, \mu_2)$, $\mathbf{x} \in \mathbb{R}^n$ and $\mu_1, \mu_2 \in \mathbb{R}$ having the following properties:

1. it has a codimension-one invariant manifold M, which is preserved under a two-parameter variation.

2. when $(\mu_1, \mu_2) = (0, 0)$ the system undergoes only a single-zero and a second order degeneracies, and

3. the one-dimensional center manifold of this singularity is transversal to the codimension-one invariant manifold.

Then this vector field undergoes a codimension-two bifurcation that involves an interaction of the saddle-node and the transcritical bifurcations.

Remark

The same degeneracy occurring in a general system gives a cusp bifurcation. The details about it can be seen in any bifurcation textbooks [47, 78].

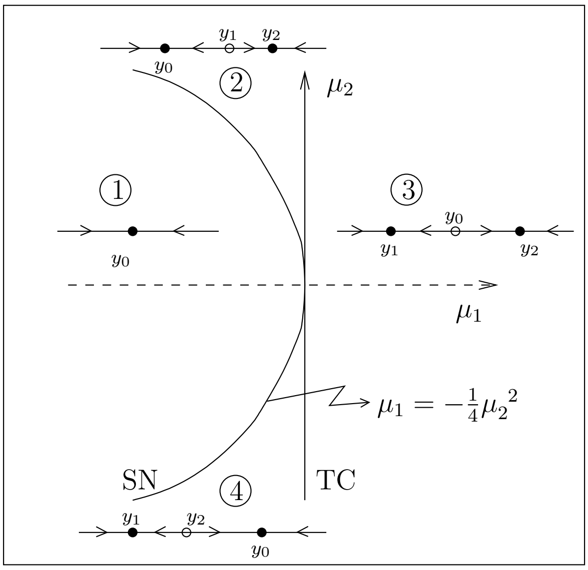

Figure 3.5: Bifurcation diagram and schematic phase portraits of (3.20) when $s = -1$. We have four topologically different areas, which are separated by saddle-node (SN) and transcritical (TC) bifurcations. The solid dots and the circle dots in each phase portraits represent asymptotically stable equilibria and unstable equilibria respectively. The full lines represent lines of bifurcation. In contrast, the dotted line just represents an axis.

3.4 Double zero eigenvalue degeneracy

3.4.1 Normal form derivation

Consider the equation (3.2). We know that a codimension-one bifurcation (i.e. the transcritical bifurcation) occurs when

$$f_n(\mathbf{0}) = 0.$$

The other degeneracy possibly comes from the matrix (3.4). This is the case in this section, where in particular we consider a double-zero eigenvalue degeneracy. The Jordan canonical form of the linear part of the system with a double-zero eigenvalue degeneracy is given by,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.29}$$

We work on a two-dimensional system since the center manifold of this degeneracy is two-dimensional, thus we have a system of two differential equations,

$$\begin{aligned} \dot{x} &= y + \mathcal{O}(\|(x,y)\|^2), \\ \dot{y} &= y(\mathcal{O}(\|(x,y)\|^1), \end{aligned} \tag{3.30}$$

where $\mathcal{O}(\|(x,y)\|^k)$ is defined below,

$$\mathcal{O}(\|(x,y)\|^k) = \sum_{i=0}^{k} \beta_{(k-i)i} x^{k-i} y^i + \mathcal{O}(\|(x,y)\|^{k+1}),$$

where k is an integer greater than zero. We shall do a normalization to find a coordinate system in which our dynamical system is as simple as possible. To start the normalization treatment of a system having a codimension-one invariant manifold, we consider the range of the operator ad $A = [.,A]$ that is spanned by these four vectors:

$$\left\{ \begin{pmatrix} 2xy \\ 0 \end{pmatrix}, \begin{pmatrix} -y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ -y^2 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \right\}. \tag{3.31}$$

These vectors are, respectively, the Lie brackets of the linear part of (3.30) with the five standard basis vectors for $H_2(\mathbb{R}^2)$, which is the space of the polynomial vector fields of degree 2, having a codimension-one invariant manifold, (note that the Lie bracket with the third term below is identically zero),

$$\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}. \tag{3.32}$$

Thus, the set of vectors below,

$$\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \right\} \tag{3.33}$$

spans a complementary subspace of the range of the operator ad A. Hence, the normal form of (3.30) can be written as:

$$\begin{aligned} \dot{x} &= y + ax^2 + \mathcal{O}(||(x,y)||^3), \\ \dot{y} &= y(bx + \mathcal{O}(||(x,y)||^2)). \end{aligned} \tag{3.34}$$

We assume that there is no other degeneracy, which means that the quadratic coefficients of the normal form above, a and b do not vanish. We initially neglect terms of order three and higher to have a two-dimensional normal form:

$$\begin{aligned} \dot{x} &= y + ax^2, \\ \dot{y} &= y(bx). \end{aligned} \tag{3.35}$$

3.4.2 Phase portrait of normal forms with a double zero degeneracy

We would like to sketch the dynamics of the system (3.35) near the origin. Using the following transformation:

$$x \mapsto v = x, \quad y \mapsto w = y + ax^2, \tag{3.36}$$

we get a new system,

$$\begin{aligned} \dot{v} &= w, \\ \dot{w} &= (2a+b)vw - abv^3, \end{aligned} \tag{3.37}$$

which is a codimension-three Bogdanov-Takens degeneracy (see Dumortier et al. [31]). The normal form of the codimension-three Bogdanov-Takens bifurcation is given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= B\xi_1\xi_2 + \epsilon_1\xi_1^3 + D\xi_1^2\xi_2 + E\xi_1^4 + \mathcal{O}(||\xi_1, \xi_2||^5) \end{aligned} \tag{3.38}$$

where $B\epsilon_1 \neq 0$. The topological dynamics near the origin of the system above (3.38) are distinguished [31] as follows:

- saddle case $\epsilon_1 > 0$, any B and D;
- focus case $\epsilon_1 < 0$ and $B^2 + 8\epsilon_1 < 0$;
- elliptic case $\epsilon_1 < 0$ and $B^2 + 8\epsilon_1 > 0$.

The bifurcation is defined to be regular if the parameters satisfy another extra condition below,

$$5\epsilon_1 D - 3BE \neq 0. \tag{3.39}$$

Applying the information above to the equation (3.37) we perform a simple calculation to get a classification for (3.35),

- saddle case $ab < 0$,
- elliptic case $ab > 0$,

where $2a + b \neq 0$ since the codimension-three Bogdanov-Takens bifurcation does not allow the coefficients of "vw" and "v^3" in the normal form (3.37) to be zero. One may notice that our bifurcation seems more degenerate due to a violation of the condition (3.39). It is mainly because we truncate the higher order terms of the normal form (3.34) that affects the topological dynamics of our vector field. We shall discuss this later on in this section.

However in our classification, we do not have the focus case. This fact is expected as we have an invariant manifold near the origin. The schematic phase portraits can be seen in Figure 3.6.

3.4.3 Local bifurcation

In our attempt to unfold this degeneracy we consider a two-parameter family which will provide all possible perturbations of the equilibrium. We now have,

$$\begin{aligned} \dot{x} &= \mu_1 + y + ax^2, \\ \dot{y} &= y(\mu_2 + bx), \end{aligned} \tag{3.40}$$

where μ_1, μ_2 are the unfolding parameters. We can do some scaling to normalize the coefficients a and b, however it does not hurt to leave them as they are for now. Without loss of generality we assume $a > 0$ as the system above is equivariant with respect to the following symmetry,

$$(x, y, \mu_1, \mu_2, a, b) \mapsto (-x, -y, -\mu_1, \mu_2, -a, -b).$$

A local analysis will show us that there are two fixed points inside the invariant manifold.

$$(x, y)_{1,2} = (\pm\sqrt{\frac{-\mu_1}{a}}, 0), \tag{3.41}$$

For the saddle case ($b < 0$) and the elliptic case ($b > 0$), when $\mu_1 < 0$ we have these two fixed points. As we vary μ_1, at the point $\mu_1 = 0$, these two fixed points collide to form a simple equilibrium, $(x, y) = (0, 0)$. Thus we have a saddle-node bifurcation curve in the parameter space defined below,

$$SN = \{(\mu_1, \mu_2) : \mu_1 = 0\}. \tag{3.42}$$

There is another fixed point which is

$$(x, y)_3 = (-\frac{\mu_2}{b}, -\mu_1 - \frac{a}{b^2}\mu_2^2). \tag{3.43}$$

As the parameters are varied, the fixed point $(x, y)_3$ coincides with one of the equilibria $(x, y)_{1,2}$ that are inside the invariant manifold and they exchange stability. Thus we find a parabola $\{(\mu_1, \mu_2) | \mu_1 + \frac{a}{b^2}\mu_2^2 = 0\}$ is a condition for the transcritical bifurcation, at which the fixed point $(x, y)_3$ coincides with one of the equilibria inside the invariant manifold and exchanges stability. The degeneracy and the non-degeneracy conditions for the saddle-node and the transcritical bifurcations above are shown in the Appendix C.

We compute the Jacobian matrix of the system (3.40),

$$J(x, y) = \begin{pmatrix} 2ax & 1 \\ by & \mu_2 + bx \end{pmatrix}. \tag{3.44}$$

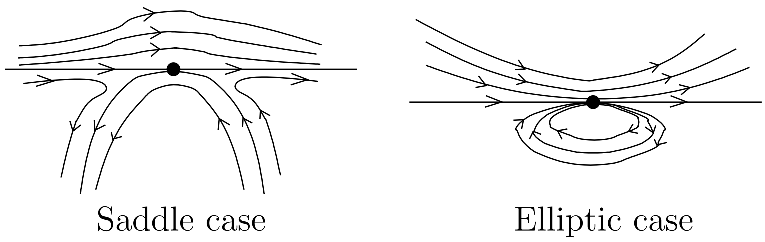

Figure 3.6: The different dynamics in the neighbourhood of the origin of the system (3.35) of the saddle case where $ab < 0$ (*left*) and of the elliptic case where $ab > 0$ (*right*)

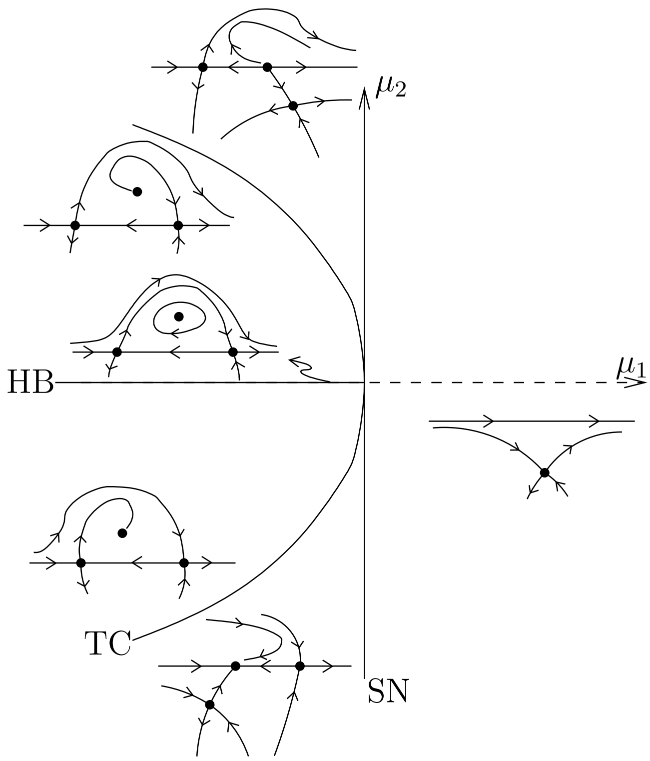

Figure 3.7: Schematic bifurcation diagram and phase portraits of the system (3.40) with μ_1 and μ_2 as parameters for the saddle case ($ab < 0$). SN, TC and HB represent saddle-node, transcritical and Hopf bifurcations respectively.

This matrix can be used to find a candidate for a Hopf bifurcation by computing at the trace and the determinant of the matrix above evaluated at the fixed point $(x, y)_3$, since the other equilibria, $(x, y)_1$ and $(x, y)_2$ are not able to undergo a Hopf bifurcation. The Hopf bifurcation is formed by equating the trace of the matrix (3.44) to zero, provided that the determinant of the same matrix is positive. These computations,

$$\begin{array}{rcccl} 0 & = & \mathrm{Tr}(J(x,y)_3) & = & -\frac{2a}{b}\mu_2, \\ 0 & < & \mathrm{Det}(J(x,y)_3) & = & b\mu_1 \end{array} \tag{3.45}$$

give conditions $\{\mu_2 = 0|\mu_1 < 0\}$ and $\{\mu_2 = 0|\mu_1 > 0\}$ for a Hopf bifurcation in the saddle and elliptic cases respectively. We now give, bifurcations sets and phase portraits for both cases (saddle and elliptic). Note that these bifurcation diagrams are partial, since we have not included a global bifurcation analysis where we may be able to see heteroclinic or homoclinic bifurcations.

The bifurcation diagram for the saddle case is depicted in Figure 3.7. First there are two bifurcation curves, which are saddle-node and transcritical bifurcations. As we cross the saddle-node line two equilibria appear and as we intersect the transcritical curve the fixed point $(x, y)_3$ that is not on the invariant axis coincides with one of the equilibria on the invariant axis and exchanges stability. There is also a Hopf bifurcation curve, at which the fixed point changes its stability. We now verify this Hopf bifurcation to show that this bifurcation is degenerate. We compute the first Lyapunov coefficient [78] of this Hopf bifurcation (using $\mu_2 = 0$). First we shall translate the equilibrium of the system (3.40) that undergoes bifurcation to the origin, using the following transformation:

$$(x, y) \mapsto (\tilde{x}, \tilde{y}) = (x/\sqrt{b\mu_1}, y + \mu_1),$$

to get a new system in the form,

$$\begin{aligned}\dot{\tilde{x}} &= \sqrt{b\mu_1}\tilde{y} + \tfrac{a}{\sqrt{b\mu_1}}\tilde{x}^2,\\ \dot{\tilde{y}} &= -\sqrt{b\mu_1}\tilde{x} + \tfrac{b}{\sqrt{b\mu_1}}\tilde{x}\tilde{y},\end{aligned}$$

where $b\mu_1 > 0$. Using the system above, we determine that the first Lyapunov coefficient of the Hopf bifurcation above is zero. This implies that the terms of at least cubic order in our normal form must be included. In fact, for $\mu_2 = 0$ (where the Hopf bifurcation occurs) the system (3.40) is completely integrable, since the function

$$F(x, y) = -by^{-\frac{2a}{b}}\left(\frac{\mu_1}{2a} + \frac{y}{2a - b} + \frac{x^2}{2}\right) \tag{3.46}$$

is constant along the solution of (3.40),

$$\begin{aligned}\dot{F} &= \tfrac{\partial F}{\partial x}\dot{x} + \tfrac{\partial F}{\partial y}\dot{y},\\ &= (-by^{-\frac{2a}{b}}x)(\mu_1 + y + ax^2) + y^{-\frac{2a}{b}}(\tfrac{\mu_1}{y} + 1 + \tfrac{ax^2}{y})(bxy),\\ &= 0.\end{aligned} \tag{3.47}$$

This integral function holds if $2a - b \neq 0$, however if this is not the case we still have an integral but it will not be of this form. This implies that when the Hopf bifurcation occurs in our system, we will have infinitely many periodic orbits and a heteroclinic link between two saddle equilibria that are living inside the codimension-one invariant manifold. This can be seen in the phase portrait of Figure 3.7, when $\mu_2 = 0$ and $\mu_1 < 0$.

The bifurcation diagram for the elliptic case is depicted in Figure 3.8. We still have curves of saddle-node and transcritical bifurcations, however a Hopf bifurcation occurs at the other side of the plane (when $\mu_2 = 0$ and $\mu_1 > 0$). The Hopf bifurcation is again degenerate since it is undetermined by the quadratic normal form. Furthermore, the system is also completely integrable with the same integral function (3.46) when $\mu_2 = 0$ at which the Hopf bifurcation occurs. We still have the fact that there are infinitely many periodic orbits, but we do not have a global bifurcation phenomenon. This permits us to conclude the local unfolding analysis of system (3.35).

We now address the effect of higher order terms in our planar system and show that some results we have "survive" while others do not. Consider the vector field (3.40) with additional higher order terms:

$$\begin{aligned}\dot{x} &= \mu_1 + y + ax^2 + \mathcal{O}(\|(x, y)\|^3),\\ \dot{y} &= y(\mu_2 + bx + \mathcal{O}(\|(x, y)\|^2)),\end{aligned} \tag{3.48}$$

By performing computations on the equations above we immediately find that the number of fixed points of the system (3.48) is more than the number of fixed points of the system without higher order terms (3.40) since the degree of the fixed point equations is higher. However, we are only interested in the neighbourhood of $(\mu_1, \mu_2) = (0, 0)$. Hence, we have the same number of equilibria involved between the system (3.40) and (3.48) locally near

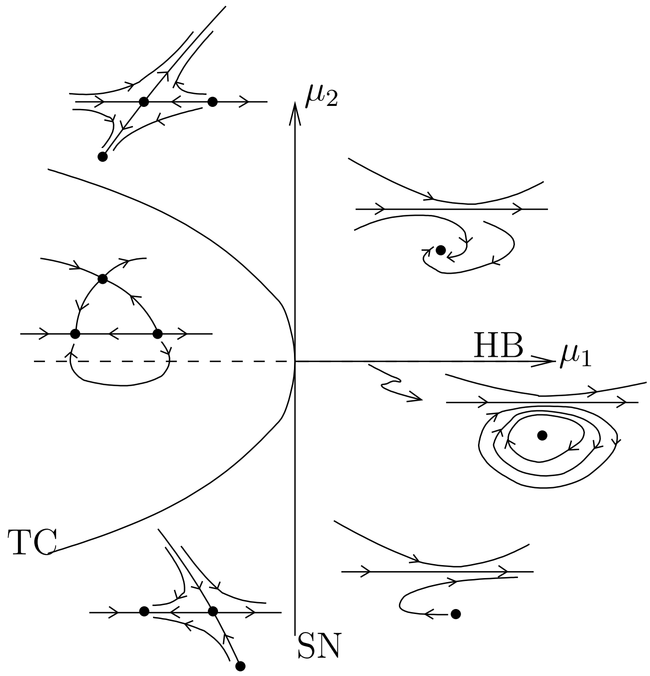

Figure 3.8: Schematic bifurcation diagram and phase portraits of the system (3.40) with μ_1 and μ_2 as parameters for the elliptic case ($ab > 0$). SN, TC and HB represent saddle-node, transcritical and Hopf bifurcations respectively.

$(\mu_1, \mu_2) = (0, 0)$. Moreover, as those fixed points near the origin are hyperbolic, they will persist for small perturbations from higher order terms as well as their stability. Then, by the Implicit Function Theorem, small perturbations of higher order terms do not significantly change the local bifurcation curves in the bifurcation diagram.

However, we now will see that the presence of higher order terms affects the local dynamics. In order to show that, we shall only add the cubic terms in the equation (3.48) as follows:

$$\begin{aligned} \dot{x} &= \mu_1 + y + ax^2 + cx^3, \\ \dot{y} &= y(\mu_2 + bx + dx^2), \end{aligned} \tag{3.49}$$

where $c, d \neq 0$. We shall show that local bifurcations will survive while some local dynamics will not.

Lemma 3.7. *The saddle-node and the transcritical bifurcations occur in the system (3.49). They are locally topologically equivalent with those of system (3.40).*

Proof. We shall first discuss the saddle-node bifurcation. The saddle-node bifurcation is obtained by the elimination of x from the two equations below,

$$\begin{aligned} \mu_1 + ax^2 + cx^3 &= 0, \\ 2ax + 3cx^2 &= 0, \end{aligned} \tag{3.50}$$

where $a, c \neq 0$. The first equation above comes from the equation that is used to find fixed points that are living inside the codimension-one invariant manifold and the latter equation is the stability equation of the fixed points. Eliminating x we get two curves in parameter space that give saddle-node bifurcations,

$$\mu_1 = 0 \quad \text{and} \quad \mu_1 = \frac{4a^3}{27c^2}. \tag{3.51}$$

However, we do not consider the bifurcation curve in the part of parameter space that is outside the neighbourhood of the origin, $(\mu_1, \mu_2) = (0, 0)$, thus we are only interested in the first saddle-node bifurcation curve, $\mu_1 = 0$. This curve, in fact is the same curve as the saddle-node bifurcation curve of system (3.40), thus we have proved the first part of this lemma.

To prove the second part of this lemma, we consider these two equations,

$$\begin{aligned} \mu_1 + ax^2 + cx^3 &= 0, \\ \mu_2 + bx + dx^2 &= 0, \end{aligned} \tag{3.52}$$

where $a, b, c, d \neq 0$. The first equation comes from the equation, used to find the condition for the fixed point to cross the invariant manifold, coincide and exchange stability with the fixed point that is inside the manifold, $y = 0$, while the second equation comes from the condition that the eigenvalue of the Jacobian matrix evaluated at the critical fixed point is zero. Eliminating x from the two equations above gives us a curve of transcritical bifurcation in the parameter space $\mu_1 - \mu_2$. We want to prove that this bifurcation curve is topologically equivalent to the transcritical bifurcation curve of system (3.40). We consider the equations above as a non-linear system of two equations with coordinates (x, μ_1, μ_2) as follows,

$$\phi(x, \mu_1, \mu_2) = \left\{ \begin{array}{lclcl} \phi_1(x, \mu_1, \mu_2) & = & \mu_2 + bx + dx^2 & = & 0, \\ \phi_1(x, \mu_1, \mu_2) & = & \mu_1 + ax^2 + cx^3 & = & 0. \end{array} \right. \tag{3.53}$$

The solution of the non-linear system above is a curve, passing through the origin since $(0, 0, 0)$ satisfies the equations above. The Jacobian matrix of the non-linear system above evaluated at the origin,

$$J(0, 0, 0) = \begin{pmatrix} b & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

has rank two since the determinant of the sub-matrix below is not zero,

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq 0.$$

In this case, the Implicit Function Theorem provides the local existence of two smooth functions, $\mu_1(x)$ and $\mu_2(x)$ for x sufficiently near the origin,

$$\mu_1(x) = -ax^2 - c^3, \quad \text{and} \quad \mu_2(x) = -bx - dx^2.$$

These functions in fact define a curve $\gamma \subset \mathbb{R}^2$ that is the transcritical bifurcation curve in the parameter space $\mu_1 - \mu_2$, parameterized by an interval of x near zero. This curve, moreover has the following characteristics: it is tangent to the saddle-node bifurcation line $\mu_1 = 0$,

$$\frac{d\mu_1}{d\mu_2}(0, 0) = \frac{\frac{d\mu_1}{dx}(0)}{\frac{d\mu_2}{dx}(0)} = 0,$$

it passes the origin $(\mu_1, \mu_2) = (0, 0)$, and the sign of μ_1-coordinate is negative for $a > 0$ for sufficiently small x near zero. These characteristics are also the case for the transcritical bifurcation of system (3.40). We now have a rough picture of this transcritical bifurcation curve. And now we want to construct a local homeomorphism of the parameter plane that maps this curve into the corresponding transcritical bifurcation curve of system (3.40) as shown in Figures 3.7 and 3.8 for saddle and elliptic cases respectively. The idea is to use the property of the parametrization of the curve γ and the fact that the transcritical curve of (3.40) can also be parameterized by the same interval. The Implicit Function Theorem guarantees that the parametrization of the curve γ is locally one-to-one. Using this property, we can construct an inverse transformation to map a point in the curve γ to the interval near $x = 0$. Finally, the image of this inverse parametrization is mapped to the parabolic curve of the transcritical bifurcation of the system (3.40). Thus, we have shown that the transcritical bifurcation curve of (3.49) is locally topologically equivalent with that of system (3.40). □

Lemma 3.8. *Consider system (3.49) with $a > 0$. A Hopf bifurcation occurs when $\mu_2 = 0$, $\mu_1 < 0$ for the saddle case and $\mu_2 = 0$, $\mu_1 > 0$ for the elliptic case. This Hopf bifurcation is non-degenerate.*

Proof. The first equation for the Hopf bifurcation is obtained by evaluating the critical equilibrium that is not on the invariant manifold. Thus $y \neq 0$ in the equation (3.49), and it implies that $\mu_2 + bx + dx^2$ must be equal to zero. We now have to compute the trace of the Jacobian matrix of system (3.49) evaluated at the critical equilibrium that undergoes a Hopf bifurcation, along with the condition that the Jacobian matrix must have a positive determinant. Hence we have the following non-linear system:

$$\begin{aligned} \mu_2 + bx + dx^2 &= 0, \\ 2ax + 3cx^2 &= 0, \\ -by - 2dxy &> 0, \end{aligned} \tag{3.54}$$

where $a, b, c, d \neq 0$. The second equation above gives us two solutions; $x = 0$ and $x = -2a/3c$. However the latter gives us a Hopf bifurcation that is far enough from the origin. Hence this is not the case that we want to discuss. Thus, we shall consider the solution $x = 0$ that gives us Hopf bifurcation conditions; $\mu_2 = 0$, $\mu_1 < 0$ for the saddle case $(b < 0)$ and $\mu_2 = 0$, $\mu_1 > 0$ for the elliptic case $(b > 0)$. We also check the non-degeneracy of the Hopf bifurcation by computing the first Lyapunov coefficient of the normal form of Hopf degeneracy. Translating the equilibrium of the system (3.49) that undergoes a Hopf bifurcation to the origin using the transformation, $(x, y) \mapsto (\tilde{x}, \tilde{y}) = (\sqrt{b\mu_1}x, y + \mu_1)$, we get a new system as follows,

$$\begin{aligned} \dot{\tilde{x}} &= \sqrt{b\mu_1}\tilde{y} + \tfrac{a}{\sqrt{b\mu_1}}\tilde{x}^2 + \tfrac{c}{b\mu_1}\tilde{x}^3, \\ \dot{\tilde{y}} &= -\sqrt{b\mu_1}\tilde{x} + \tfrac{b}{\sqrt{b\mu_1}}\tilde{x}\tilde{y} - \tfrac{d}{b}\tilde{x}^2 + \tfrac{d}{b\mu_1}\tilde{x}^2\tilde{y}, \end{aligned}$$

where $b\mu_1 > 0$. Using the system above, we obtain the first Lyapunov coefficient,

$$l_1 = \frac{1}{8\omega_0^3}(6c + 2d) - \frac{2\mu_1 d}{4\omega_0^5}(b + 2a), \qquad \omega_0 = -\sqrt{b\mu_1}, \tag{3.55}$$

that is generally non-zero for every cubic term. Hence, the Hopf bifurcation of system (3.49) is non-degenerate. □

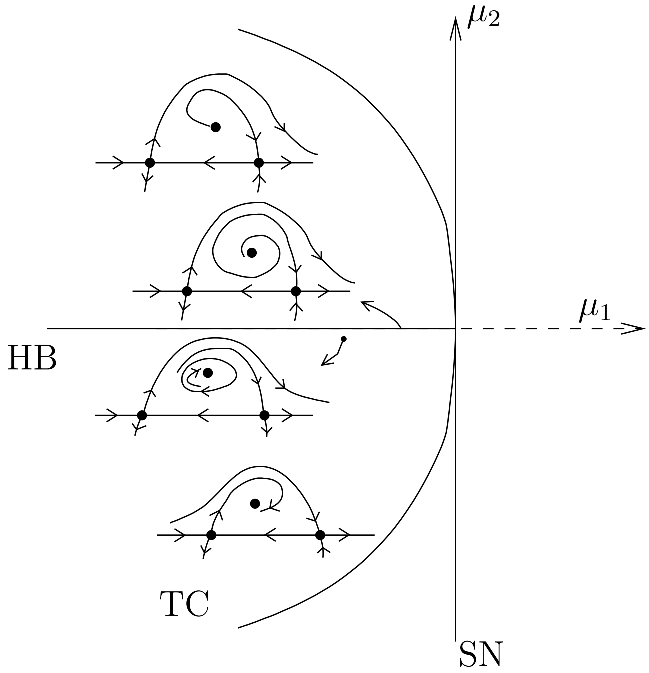

Figure 3.9: Phase portraits and partial bifurcation diagram of the system (3.49) with non-zero c and d for the saddle case ($ab < 0$). SN, TC and HB represent saddle-node, transcritical and Hopf bifurcations respectively. This bifurcation diagram is not topologically equivalent with the bifurcation diagram of the saddle case with $c = d = 0$, presented in Figure 3.7, especially in the area slightly below the Hopf bifurcation line. There is a new phase portrait that does not occur in the previous saddle case.

Thus, we have shown that local bifurcations such as transcritical, saddle-node and Hopf persist for small perturbations of cubic terms. However, we notice that the Hopf bifurcation is now non-degenerate as a result of cubic order terms. In the next section, we will show that the addition of cubic terms will change the local dynamics and give birth to a global bifurcation.

3.4.4 Global bifurcation

From the two lemmas above we conclude that the basic local bifurcations survive. Now we shall see that the presence of cubic terms affects the local dynamics. We shall take an example of the saddle case where we include the cubic terms. We assume that the sign of the coefficients of the cubic terms are both positive. We draw a bifurcation diagram and the phase portraits corresponding to system (3.49) in Figure 3.9. Comparing it with Figure 3.7, we see that the fixed point that undergoes a Hopf bifurcation is unstable when the parameters are above the Hopf bifurcation line. However, we see a significant difference between the two figures. When the parameters are exactly at the Hopf line, we have two different phase portraits. In Figure 3.7 the stability of the fixed point is undetermined, however in the system with higher order terms (see Figure 3.9) the fixed point is unstable. We also see a new phase portrait that we never saw before when the parameters are slightly below the Hopf line . The fixed point is stable, which agrees with our analysis, however there is an unstable limit cycle in the system with higher order terms. Hence, there must be an additional global bifurcation curve

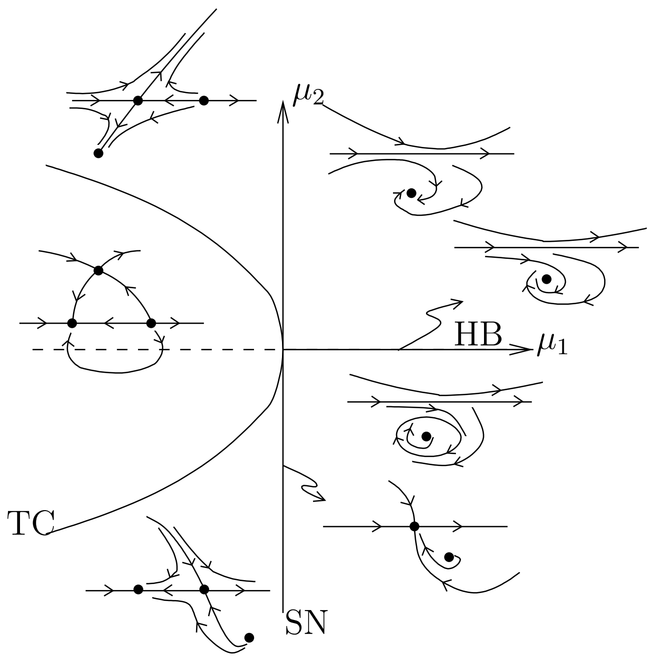

Figure 3.10: Bifurcation diagram of the system (3.49) with non-zero c and d for the elliptic case ($ab > 0$). SN, TC and HB represent saddle-node, transcritical and Hopf bifurcations respectively.

in this area (below the Hopf line) since this phase portrait is not homeomorphic with the other phase portrait from the same area. The saddle equilibria that are inside the codimension-one invariant manifold and the equilibrium that undergoes Hopf bifurcation do not change their topological types, and thus a global bifurcation must take place. We have to note that the significant differences of the phase portraits depend on the signs of the cubic terms, c and d. If we change the sign of either c or d then the occurrence of limit cycle may happen in the area above the Hopf bifurcation line.

On the other hand, for the elliptic case, the significant difference between (3.35) and (3.49) is the existence of an isolated limit cycle in the area below the Hopf bifurcation line in the latter case (compare Figures 3.8 and 3.10). We assume that the signs of the coefficients of the cubic terms are both negative. When the parameters are in that area, the phase portrait always has a stable limit cycle. This cycle collapses when the parameters cross the saddle-node bifurcation, $\mu_1 = 0$. Hence, different from the saddle case we do not expect a global bifurcation taking place in this diagram.

We go back the the saddle-case to locate a global bifurcation taking place in our diagram, we let $a = 1$ and $b = -1$. We rescale the coordinate along with the unfolding parameters as follows,

$$x = \epsilon u, \quad y = \epsilon^2 v, \quad \mu_1 = \epsilon^2 \alpha_1, \quad \mu_2 = \epsilon^2 \alpha_2, \tag{3.56}$$

and rescale the time $t \mapsto \epsilon t$, so that (3.49) becomes

$$\begin{aligned} \dot{u} &= \alpha_1 + v + au^2 + \epsilon c u^3, \\ \dot{v} &= \epsilon \alpha_2 v + buv + \epsilon d u^2 v. \end{aligned} \tag{3.57}$$

Note that for $\epsilon = 0$ we have a system that is completely integrable, with an integral as follows,

$$F(u,v) = -bv^{-\frac{2a}{b}}\left(\frac{\alpha_1}{2a} + \frac{v}{2a-b} + \frac{u^2}{2}\right).$$

We take an example of the saddle case, where $a = 1$ and $b = -1$. Furthermore, without loss of generality we set $\alpha_1 = -1$ since the case of interest occurs when $\mu_1 < 0$. The variation of $\mu_1 < 0$ is obtained as ϵ varies. We now consider the system (3.57) multiplied by the integrating factor v^{l-1} where $l = -2a/b$,

$$\begin{aligned} \dot{u} &= \alpha_1 v^{l-1} + v^l + au^2v^{l-1} + \epsilon cu^3v^{l-1}, \\ \dot{v} &= buv^l + \epsilon\alpha_2 v^l + \epsilon du^2v^l. \end{aligned} \tag{3.58}$$

The above system is a dilated version of the vector field (3.57) for $v > 0$, thus the solution curves of (3.57) are topologically equivalent to those of (3.58). We would like to show that for small ϵ and suitable choices of (α_2, c, d), the isolated level curve (i.e. heteroclinic orbit) is preserved. First we set the system above in a vector notation,

$$\dot{\mathbf{w}} = \mathbf{k}(\mathbf{w}) + \epsilon\mathbf{l}(\mathbf{w}, \alpha_2). \tag{3.59}$$

Applying the Melnikov method and Green's theorem, we deduce that given a closed curve Γ we have

$$\int_{\text{int }\Gamma} \text{trace } D\mathbf{l}(\mathbf{w}, \alpha_2)\ \mathrm{d}\mathbf{w} = 0, \tag{3.60}$$

for a chosen α_2 where the trace of $D\mathbf{l}$ is given by

$$\text{trace } D\mathbf{l} = v^{l-1}(3cu^2 + l\alpha_2 + dlu^2); \quad l = -2a/b. \tag{3.61}$$

We automatically have trace $D\mathbf{k} = 0$, since it is integrable. We therefore must find a value, K such that $F^{-1}(K)$ is a heteroclinic curve Γ_K. It turns out that the value $K = 0$ corresponds to the heteroclinic orbit. Gathering all the information above and the facts that we have $a = 1$, $b = -1$ and $\alpha_1 = -1$, we have to integrate

$$\int\int_{\text{int }\Gamma_K} [(3c + 2d)u^2v + 2\alpha_2 v]du\, dv, \tag{3.62}$$

where the closed curve Γ_K is given by

$$v^2\left(-\frac{1}{2} + \frac{v}{3} + \frac{u^2}{2}\right) = 0. \tag{3.63}$$

Evaluating the integral above gives

$$\frac{12}{5}\alpha_2 + \frac{18}{35}c + \frac{12}{35}d = 0. \tag{3.64}$$

This equation determines the location of the heteroclinic bifurcation in our parameter space (up to order ϵ), and in terms of the variables used before scaling, we obtain the equation for the heteroclinic bifurcation curve:

$$\mu_2 = \frac{(3c + 2d)}{14}\mu_1 + \mathcal{O}(\epsilon). \tag{3.65}$$

We have proven the following lemma.

Lemma 3.9. *There is a curve in the bifurcation diagram of system (3.49), corresponding to a heteroclinic bifurcation and having the following representation,*

$$\{(\mu_1, \mu_2) : \mu_2 \approx \frac{(3c + 2d)}{14}\mu_1\}$$

Thus, the complete bifurcation diagrams of (3.49) for the saddle and the elliptic cases are depicted in Figure 3.11 and 3.10 respectively.

Remark

- Double-zero degeneracy occuring in a general system (i.e. systems without a special structure) gives us a codimension-two Bogdanov-Takens bifurcation, which has been studied in great detail [47, chapter 7].
- We call this bifurcation a second interaction of saddle-node and transcritical bifurcations. The first interaction of saddle-node and transcritical bifurcations occurs in the previous section where we have a single-zero degeneracy combined with a second order degeneracy.
- The operator ad $A = [\ .\ , A]$ that we used above is explained in detail in Guckenheimer and Holmes [47, chapter 3] and also in Broer et al. [12, chapter 6]
- The topological classification and the unfolding of the degenerate Bogdanov-Takens bifurcation of codimension three have been completely analysed by Dumortier et al. [31] while the computation of the normal form of the general system with a codimension-three Bogdanov-Takens bifurcation has been obtained by Kuznetsov [79].
- The theory of Melnikov method that is used to locate a global bifurcation can be seen in any dynamical textbook, for instance Guckenheimer and Holmes [47, chapter 4].

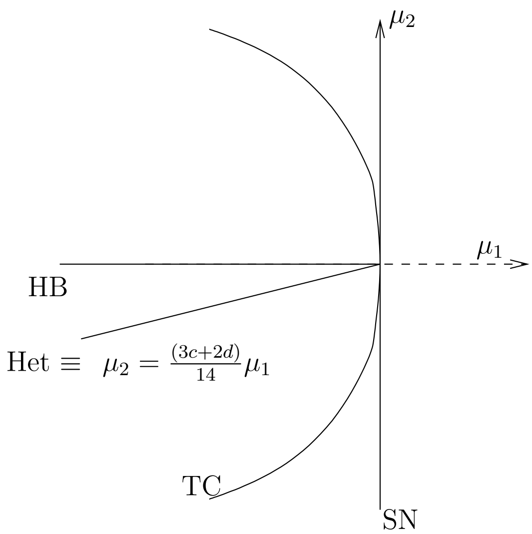

Figure 3.11: Complete bifurcation diagram of the system (3.49) for the saddle case. SN, TC and HB represent saddle-node, transcritical and Hopf bifurcations respectively, while Het is a heteroclinic bifurcation.

3.5 A single-zero and a pair of purely imaginary eigenvalues

In this section, we provide an analysis of the remaining bifurcation of codimension-two that occurs in a system having a codimension-one invariant manifold. We discuss the problem of a single-zero and a pair of purely imaginary eigenvalues degeneracies. We work on a three-dimensional system since we can reduce the dimension of the system by a center manifold reduction. When we deal with a pair of purely imaginary eigenvalues, it is always convenient to work in polar coordinates, which we will do in a moment. Moreover, we will see that we can reduce the three-dimensional system into a two-dimensional system by removing the angle part of our system under some assumptions. Thus, most of the analyse in this section are mainly planar. We will translate some of the results we obtain in the planar analysis to the three dimensional system. Some complex dynamics shall appear since some assumptions that previously applied do not apply anymore.

The Jordan canonical form of the linear part of our system will be:

$$A = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.66}$$

Thus we have a three-dimensional system as follows,

$$\begin{aligned} \dot{x}_1 &= -\omega x_2 + \mathcal{O}(\|(x_1, x_2, y)\|^2), \\ \dot{x}_2 &= \omega x_1 + \mathcal{O}(\|(x_1, x_2, y)\|^2), \\ \dot{y} &= y(\mathcal{O}(\|(x_1, x_2, y)\|)). \end{aligned} \tag{3.67}$$

Using the method of normal form, we can remove some nonlinear terms in the equation above. Moreover, the normal form can be conveniently written in the cylindrical polar coordinates as follows,

$$\begin{aligned} \dot{r} &= a_1 r y + a_2 r^3 + a_3 r y^2 + \mathcal{O}(\|(r, y)\|^4), \\ \dot{\theta} &= \omega + \mathcal{O}(\|(r, y)\|^2), \\ \dot{y} &= y(b_1 y + b_2 r^2 + b_3 y^2 + \mathcal{O}(\|(r, y)\|^3)), \end{aligned} \tag{3.68}$$

provided that all the normal form coefficients are non-zero. It turns out that the θ-dependence in the r and y components of the vector field can be removed to order k for k arbitrarily large. This is important since we can truncate our equation to some order and ignore the $\dot{\theta}$ part of our vector field. Then we perform a local bifurcation analysis on the r, y parts of the vector field. In some sense, for r, y sufficiently small, the $r - y$ plane can be thought of as a Poincaré map for the full three-dimensional system. We thus remove the $\dot{\theta}$ part and truncate terms of order four and higher,

$$\begin{aligned} \dot{r} &= a_1 r y + a_2 r^3 + a_3 r y^2, \\ \dot{y} &= y(b_1 y + b_2 r^2 + b_3 y^2). \end{aligned} \tag{3.69}$$

Before we do further analysis on the system above we shall do another transformation that helps reduce the number of parameters we have. We introduce a new coordinate by the following transformation:

$$\begin{aligned} s &= r(1 + gy), \\ w &= y + hr^2 + iy^2, \\ \tau &= (1 + jy)^{-1} t, \end{aligned} \tag{3.70}$$

and compute the vector field (3.69) with respect to the new coordinate:

$$\begin{aligned}\frac{ds}{d\tau} &= a_1sw + (a_2 - a_1h)s^3 + (a_3 + b_1g - a_1i + a_1j)sw^2 + R_s(s,w),\\ \frac{dw}{d\tau} &= b_1w^2 + (b_2 + 2a_1h - 2b_1h)s^2w + (b_3 + b_1j)w^3 + R_w(s,w).\end{aligned} \tag{3.71}$$

The remainder terms have order at least four in (s,w), hence we ignore these higher order terms. We now choose (g,h,i,j) to make (3.71) as simple as possible. The new cubic coefficients introduced in the above system depend linearly on (g,h,i,j) as described by the matrix:

$$\mathbf{M} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -b_1 & 0 & a_1 & -a_1 \\ 0 & -2a_1 + 2b_1 & 0 & 0 \\ 0 & 0 & 0 & -b_1 \end{pmatrix}, \tag{3.72}$$

where $\mathbf{v}$ is (g,h,i,j) and $\mathbf{Mv}$ has the components $(-a_1h, -b_1g + a_1i - a_1j, -2a_1h + 2b_1h, -b_1j)$ which is added to the components s^3, sw^2, s^2w and w^3. The matrix has rank three with a kernel spanned by the vector $(a_1, 0, b_1, 0)$. Consequently, we may choose $\mathbf{v}$ such that $\mathbf{Mv}$ is equal to $(0, a_3, 0, b_3)$ since this vector is in the range of $\mathbf{M}$. Thus, we can assume that the coefficients of sw^2 and w^3 are zero and consider only the cubic perturbation (s^3, s^2w) as follows:

$$\begin{aligned}\dot{r} &= a_1ry + a_2r^3,\\ \dot{y} &= y(b_1y + b_2r^2).\end{aligned} \tag{3.73}$$

3.5.1 Phase portraits of normal forms with a Hopf-zero degeneracy

We consider the equation (3.73) and scale the above system by the following transformation:

$$\bar{r} = \alpha r \quad \text{and} \quad \bar{y} = \beta y. \tag{3.74}$$

Then the equation (3.73) becomes

$$\begin{aligned}\dot{\bar{r}} &= \alpha(a_1\tfrac{\bar{r}\bar{y}}{\alpha\beta} + a_2\tfrac{\bar{r}^3}{\alpha^3}),\\ \dot{\bar{y}} &= \bar{y}(b_1\tfrac{\bar{y}}{\beta} + b_2\tfrac{\bar{r}^2}{\alpha^2}.\end{aligned} \tag{3.75}$$

We set $\beta = -b_1$ and $\alpha = \sqrt{|b_2|}$ and drop the bars, thus it yields

$$\begin{aligned}\dot{r} &= a_1ry + a_2r^3,\\ \dot{y} &= y(-y + sr^2),\end{aligned} \tag{3.76}$$

where $s = \pm 1$. The coefficients, a_1, a_2 are different from those of the equation (3.73), however we keep the same notations for convenience. They can be positive or negative, but will be assumed to be non-zero. We also need $a_1 - a_2 \neq 0$ as will be explained below. We note that we have two invariant manifolds in this case, one is $y = 0$ (our codimension-one invariant manifold) and one is $r = 0$ as a result of the symmetry $(r,y) \mapsto (-r,y)$. Thus we only need to consider half the $r - y$ plane due to this symmetry. We also have another symmetry that involves some parameters which is

$$(s, y, t, a_2) \mapsto -(s, y, t, a_2). \tag{3.77}$$

As a consequence, we can set $s = -1$ without further considering the case $s = 1$ since it follows from the symmetry above.

We now want to classify the phase portraits of (3.76) near the origin. Consider the $r-y$ half plane, $r \geq 0$. We shall divide cases here, firstly we blow the area $y > 0$ up and then the area $y < 0$. In the area $y > 0$, we perform the first blowing-up:

$$(r, y) \mapsto (R, Y^2), \tag{3.78}$$

leading to

$$\begin{array}{rcl} \dot{R} & = & a_1 R Y^2 + a_2 R^3, \\ \dot{Y} & = & -\frac{1}{2}Y^3 - \frac{1}{2}R^2 Y. \end{array} \tag{3.79}$$

By means of polar blowing-up $R = \rho\cos\theta$ and $Y = \rho\sin\theta$, we get:

$$\begin{array}{rcl} \dot{\rho} & = & \rho(a_1\cos^2\theta\sin^2\theta + a_2\cos^4\theta - \frac{1}{2}\sin^4\theta - \frac{1}{2}\cos^2\theta\sin^2\theta), \\ \dot{\theta} & = & -\cos\theta\sin\theta((a_1+\frac{1}{2})\sin^2\theta + (a_2+\frac{1}{2})\cos^2\theta), \end{array} \tag{3.80}$$

where $\rho \geq 0$ and $\theta \in [0, \frac{\pi}{2}]$.

There are three possible equilibria on $\rho = 0$, which are:

(i) $\rho = 0$, $\theta = 0$, whose linearization matrix is:

$$J(0,0) = \begin{pmatrix} a_2 & 0 \\ 0 & -a_2 - \frac{1}{2} \end{pmatrix},$$

(ii) $\rho = 0$, $\theta = \frac{\pi}{2}$, whose linearization matrix is:

$$J(0, \frac{\pi}{2}) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & a_1 + \frac{1}{2} \end{pmatrix},$$

(iii) $\rho = 0$, $\theta = \alpha$, where $\alpha = \arctan\sqrt{-\frac{a_1+\frac{1}{2}}{a_2+\frac{1}{2}}}$. The Jacobian matrix evaluated at this equilibrium is

$$J(0,\alpha) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{(2a_1+1)(2a_2+1)}{2(a_1-a_2)} \end{pmatrix}.$$

Hence there are six areas in the $a_1 - a_2$ plane that will yield qualitatively different phase portraits of (3.80) which are

1. $a_1 > -\frac{1}{2}$,$a_2 > 0$,
2. $a_1 > -\frac{1}{2}$,$-\frac{1}{2} < a_2 < 0$,
3. $a_1 > -\frac{1}{2}$,$a_2 \leq -\frac{1}{2}$,
4. $a_1 \leq -\frac{1}{2}$,$a_2 > 0$,
5. $a_1 \leq -\frac{1}{2}$,$-\frac{1}{2} < a_2 < 0$ and
6. $a_1 \leq -\frac{1}{2}$,$a_2 \leq -\frac{1}{2}$.

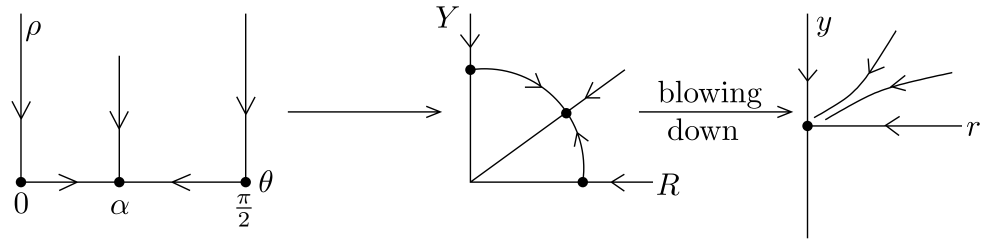

Figure 3.12: The blowing-up method to analyse the phase portrait of (3.76). We take an example of area 3 where $a_1 > -\frac{1}{2}$ and $a_2 \leq -\frac{1}{2}$.

We note that the third equilibrium $(0, \alpha)$ does not appear in the area 1, 2 and 6. Performing the phase portrait analysis and the blowing-down transformation we get six qualitatively different phase portraits near the origin of (3.73). We take an example that is illustrated in Figure 3.12. We choose the area 3. Using all the computations above we know that the equilibria $(\theta = 0)$ and $(\theta = \frac{\pi}{2})$ are stable in the $\rho-$direction and unstable in the $\theta-$direction. In this area, the equilibrium $(\theta = \alpha)$ also appears and is stable in all directions. We then do the blowing-down transformation to have the phase portrait for this area.

We now blow the second area $(y < 0)$ up. We perform the following transformation:

$$(r, y) \mapsto (R, -Y^2). \tag{3.81}$$

Then we get,

$$\begin{aligned} \dot{R} &= -a_1 R Y^2 + a_2 R^3, \\ \dot{Y} &= \tfrac{1}{2} Y^3 - \tfrac{1}{2} R^2 Y. \end{aligned} \tag{3.82}$$

We then do the second blowing-up which is the polar one; $R = \rho \cos\theta$ and $Y = \rho \sin\theta$ to get:

$$\begin{aligned} \dot{\rho} &= \rho(-a_1 \cos^2\theta \sin^2\theta + a_2 \cos^4\theta + \tfrac{1}{2}\sin^4\theta - \tfrac{1}{2}\cos^2\theta \sin^2\theta), \\ \dot{\theta} &= -\cos\theta \sin\theta((a_1 + \tfrac{1}{2})\sin^2\theta - (a_2 + \tfrac{1}{2})\cos^2\theta), \end{aligned} \tag{3.83}$$

where $\rho \geq 0$ and $\theta \in [0, \frac{\pi}{2}]$. Again, on $\rho = 0$ there are three possible equilibria which are $(0, 0)$, $(0, \frac{\pi}{2})$ and $(0, \alpha)$, where α is now equal to $\arctan\sqrt{\frac{a_1+\frac{1}{2}}{a_2+\frac{1}{2}}}$. The correspondence Jacobian matrices for these three equilibria are:

$$J(0,0) = \begin{pmatrix} a_2 & 0 \\ 0 & -a_2 - \frac{1}{2} \end{pmatrix}, \quad J(0, \frac{\pi}{2}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -a_1 - \frac{1}{2} \end{pmatrix},$$

and

$$J(0, \alpha) = \begin{pmatrix} -\frac{a_1 - a_2}{2(a_1+a_2+1)} & 0 \\ 0 & \frac{(2a_1+1)(2a_2+1)}{2(a_1+a_2+1)} \end{pmatrix}.$$

As the previous blowing-up above, we also have six different areas in the $a_1 - a_2$ plane. However, we need to consider the sign of $(a_1 - a_2)$ now since the third equilibrium appears in the areas 1,2 and 6 where the sign of the term $(a_1 - a_2)$ comes into play. Thus, we divide areas 1,2 and 6 considering the sign of $(a_1 - a_2)$,

(1a) $a_1 > -\frac{1}{2}$,$a_2 > 0$,$a_1 - a_2 > 0$

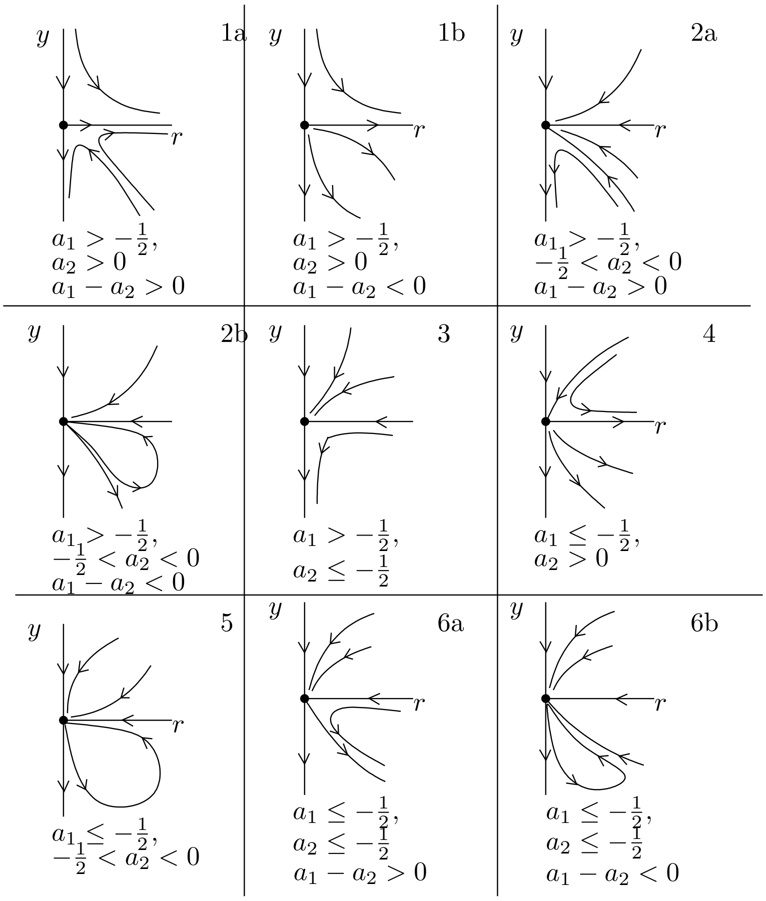

Figure 3.13: Phase portraits of (3.76) for different values of a_1 and a_2

(1b) $a_1 > -\frac{1}{2}$,$a_2 > 0$,$a_1 - a_2 < 0$

(2a) $a_1 > -\frac{1}{2}$,$-\frac{1}{2} < a_2 < 0$,$a_1 - a_2 > 0$,

(2b) $a_1 > -\frac{1}{2}$,$-\frac{1}{2} < a_2 < 0$,$a_1 - a_2 < 0$

(6b) $a_1 \leq -\frac{1}{2}$,$a_2 \leq -\frac{1}{2}$,$a_1 - a_2 > 0$ and

(6b) $a_1 \leq -\frac{1}{2}$,$a_2 \leq -\frac{1}{2}$,$a_1 - a_2 < 0$.

So we have a total of nine qualitatively different phase portraits for $y < 0$. Combined with the case $y > 0$ we obtain the qualitative phase portraits for the singularity of a single-zero eigenvalue and a pair of purely imaginary eigenvalue in the system having a codimension-one invariant manifold for different values of a_1 and a_2 as represented in Figure 3.13.

3.5.2 Bifurcation analysis of Hopf-zero normal forms

We now study the unfolding of (3.76) with respect to the nine cases we have got above. First we require that the symmetry $(r, y) \mapsto (-r, y)$ and the codimension-one invariant manifold are preserved under the perturbation of parameters. The local unfolding of this singularity is given by:

$$\begin{aligned} \dot{r} &= \mu_1 r + a_1 r y + a_2 r^3, \\ \dot{y} &= y(\mu_2 - y - r^2). \end{aligned} \tag{3.84}$$

We immediately notice that there are possibly four equilibria;

- $E_0 = (r_0, y_0) = (0, 0)$,
- $E_1 = (r_1, y_1) = (0, \mu_2)$,
- $E_2 = (r_2, y_2) = (\sqrt{-\frac{\mu_1}{a_2}}, 0)$, for $-\frac{\mu_1}{a_2} \geq 0$ and
- $E_3 = (r_3, y_3) = (\sqrt{\frac{\mu_1+a_1\mu_2}{a_1-a_2}}, -\frac{\mu_1+a_2\mu_2}{a_1-a_2})$ for $\frac{\mu_1+a_1\mu_2}{a_1-a_2} \geq 0$.

Note that the fixed point E_0 is the origin, the fixed point E_1 is the one on the $y-$axis and the fixed point E_2 is the one on the $r-$axis.

Thus, we have a line $\{(\mu_1, \mu_2)|\mu_2 = 0\}$ which is a transcritical bifurcation, which is a condition for fixed points E_0 and E_1 to coincide and exchange their stabilities. Another bifurcation line is a pitchfork bifurcation line $\{(\mu_1, \mu_2)|\mu_1 = 0\}$. When the sign of $-\frac{\mu_1}{a_2}$ is positive there appears an equilibrium E_2 on the invariant manifold, $y = 0$. Another equilibrium, E_3 also appears through the secondary pitchfork bifurcation, which occurs on the bifurcation line, $\{(\mu_1, \mu_2)|\mu_1 + a_1\mu_2 = 0\}$. When $\frac{\mu_1+a_1\mu_2}{a_1-a_2}$ is greater than zero, this equilibrium appears. Finally, the transcritical bifurcation between the fixed points E_2 and E_3 occurs on the following line, $\{(\mu_1, \mu_2)|\mu_1 + a_2\mu_2 = 0\}$, provided that the sign of $-\frac{\mu_1}{a_2}$ and $\frac{\mu_1+a_1\mu_2}{a_1-a_2}$ are both positive.

Up to this point, we have already four bifurcation lines for arbitrary values of a_1 and a_2. The behaviour of the phase portraits in all cases are also relatively simple since we have not considered Hopf bifurcations. Among all the equilibria, E_3 is the only fixed point that can undergo a Hopf bifurcation. This is mainly because the eigenvalues of the linearization matrices evaluated at E_0, E_1, and E_2 are always real. The linearization matrix evaluated at the equilibrium E_3 is:

$$J(E_3) = \begin{pmatrix} 2a_2r_3^2 & a_1y_3 \\ -2r_3y_3 & -y_3 \end{pmatrix}. \tag{3.85}$$

with the trace,

$$\text{trace of } J(E_3) = 2((a_2 + \frac{1}{2})\mu_1 + 2a_2(a_1 + \frac{1}{2})\mu_2)(a_1 - a_2)^{-1}, \tag{3.86}$$

and the determinant,

$$\det(J(E_3)) = -2(\mu_1 + a_1\mu_2)(\mu_1 + a_2\mu_2)(a_1 - a_2)^{-1}. \tag{3.87}$$

To detect a Hopf bifurcation we need the trace of (3.85) to be zero and the determinant of (3.85) to be positive. It turns out that a Hopf bifurcation cannot occur in cases 1b, 2a, 3, 4 and 6a. This is mainly because the Hopf bifurcation line (equation (3.86) is equal to zero) lies in the area where the sign of the determinant (3.87) is negative. On the other hand, a Hopf bifurcation occurs in cases 1a, 2b, 5 and 6b.

We will now describe these bifurcations in bifurcation diagrams. We recall that, we have nine qualitatively different phase portraits when $\mu_1 = \mu_2 = 0$, and that came from dividing the $a_1 - a_2$ plane into nine regions by these four lines: $a_1 = -\frac{1}{2}$, $a_2 = -\frac{1}{2}$, $a_2 = 0$ and $a_1 - a_2 = 0$. However, when unfolding system (3.84), we can reduce the number of cases that we have. This can be done mainly because the equations $a_1 = -\frac{1}{2}$ and $a_2 = -\frac{1}{2}$ do not play roles in the unfolding program as they do not affect the stability of the four equilibria when we cross these lines, $a_1 = -\frac{1}{2}$ and $a_2 = -\frac{1}{2}$. We only require that a_1, a_2 and $a_1 - a_2$ are not zero. Thus, there are only four cases that will be considered here, which are

(I) $a_2 > 0$ and $a_1 - a_2 > 0$,

(II) $a_2 > 0$ and $a_1 - a_2 < 0$

(III) $a_2 < 0$ and $a_1 - a_2 > 0$, and

(IV) $a_2 < 0$ and $a_1 - a_2 < 0$.

As a result cases 1b and 4 from the first classification in Figure 3.12 can be combined into one unfolding (case (II)) since their unfolding are not distinguishable. This is also true for cases 2a, 3 and 6a that are combined in case (III) and also cases 2b, 5 and 6b in case (IV).

We begin with the unfolding of case (I). As analysed above, in this case we have two transcritical bifurcation lines and two pitchfork bifurcation lines along with a Hopf bifurcation curve. We can see the schematic bifurcation diagram and the schematic phase portraits in each area in Figure 3.14. However, the normal form of (3.84) does not suffice to determine the type of the Hopf bifurcation in this case as we see in this figure. The dynamics of the system above the Hopf line changes quite drastically when we cross the Hopf line. When parameters lie on the Hopf bifurcation line, we have an integrable-like phase portrait in which we see infinitely many periodic solutions.

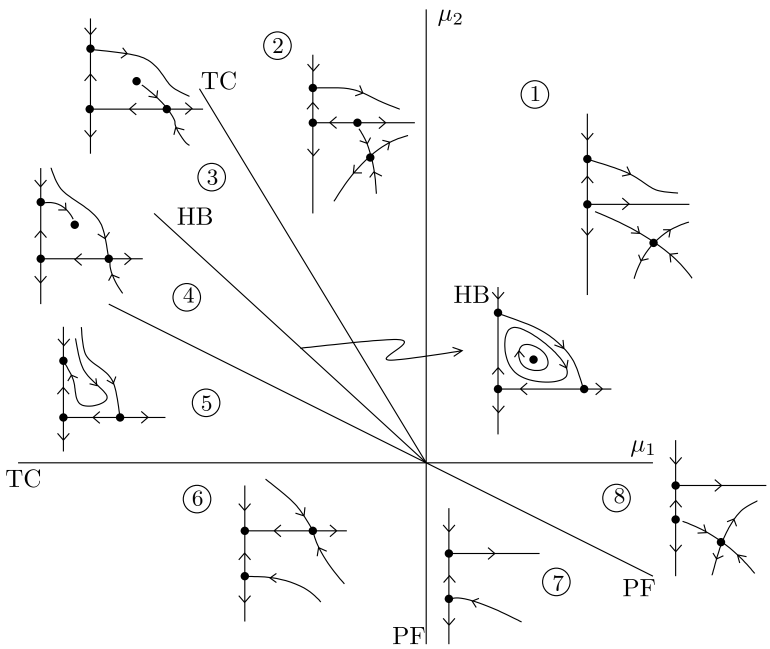

Figure 3.14: Bifurcation diagram of the system (3.84) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (I) of the unfolding program (or case 1a in the classification of (3.76)). TC, PF and HB represent transcritical, pitchfork and Hopf bifurcations respectively.

We now turn to cases (II) and (III). These unfoldings are relatively simple since these cases (1b, 2a, 3, 4 and 6a) do not undergo Hopf bifurcation in their unfoldings. One can see the bifurcation diagrams for these cases in Figure 3.15 and Figure 3.16 respectively. These bifurcation diagrams are depicted by assuming $a_1 > -\frac{1}{2}, a_2 > 0, a_1 - a_2 < 0$ for case (II) and $a_1 - > \frac{1}{2}, a_2 < 0, a_1 - a_2 > 0$ for case (III).

Our final case, case (IV) which includes cases 2b, 5 and 6b in the classification of (3.76), is rather complicated, as a Hopf bifurcation plays a role here. As noticed in the previous analysis in case (I), the Hopf bifurcation in this case is degenerate. One can see that the equilibrium E_3 drastically changes from a stable fixed point to an unstable fixed point. The dynamics around it also change significantly. It is mainly because, when the parameters μ_1 and μ_2 are on the Hopf bifurcation line, we have an integrable-like phase portrait where we see infinitely many periodic solutions. This problem will disappear if we add other normal form coefficients. We note that the fact that the Hopf bifurcation line lies inside the second quadrant ($\mu_1 < 0, \mu_2 > 0$) is because we depict this schematic bifurcation diagram by assuming $a_1 < -\frac{1}{2}, a_2 > -\frac{1}{2}$. If we vary a_1 and a_2, provided that we are still inside case (IV), we shall have all the bifurcations that we had before, however the Hopf bifurcation will take place in a different quadrant in the parameter space.

We now want to consider the effect of higher order terms. As those fixed points near the origin are hyperbolic, they will persist for small perturbations from higher order terms as

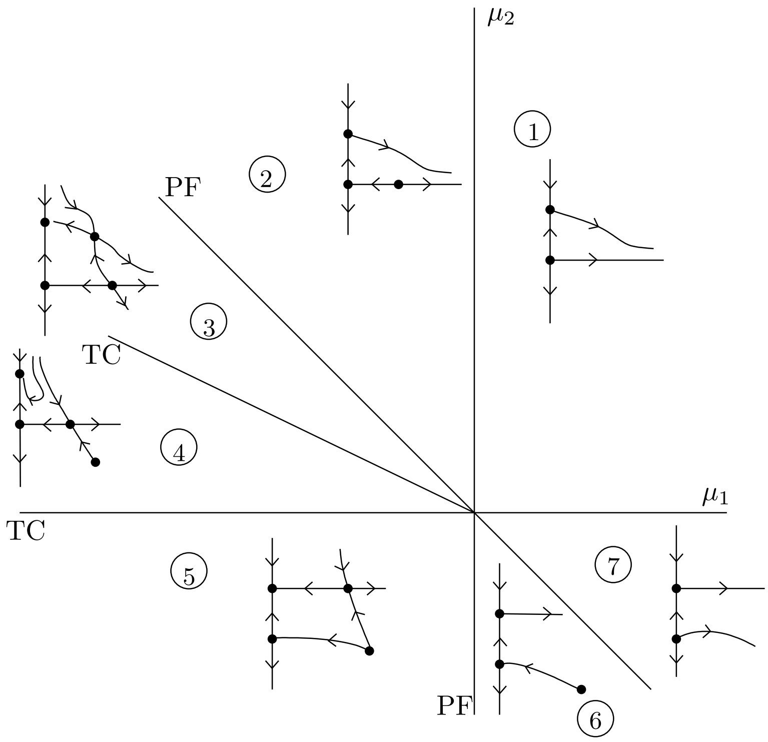

Figure 3.15: Bifurcation diagram of the system (3.84) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (II) of the unfolding program (or case 1b and 4 in the classification of (3.76)). TC and PF represent transcritical and pitchfork bifurcations respectively.

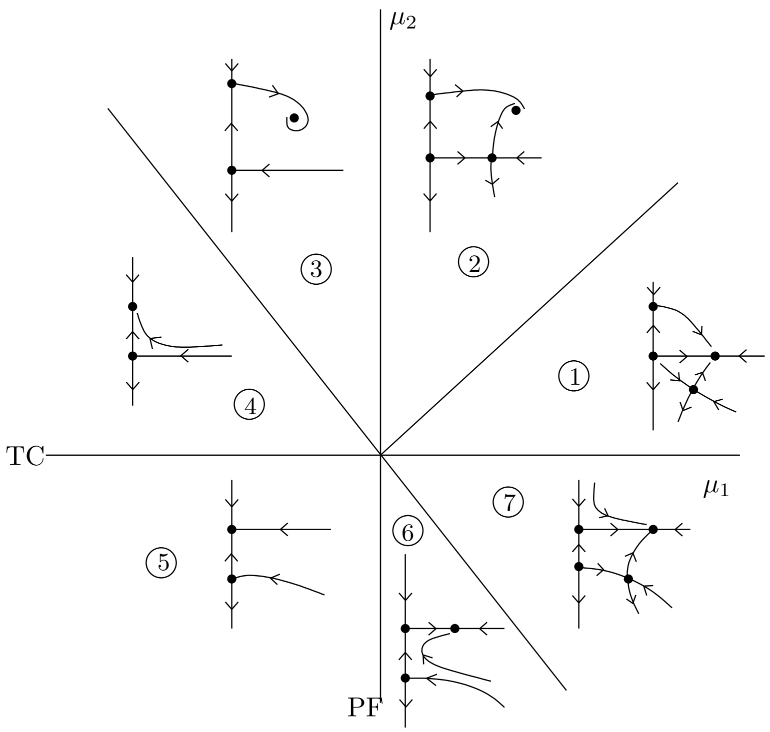

Figure 3.16: Bifurcation diagram of the system (3.84) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (III) of the unfolding program (or case 2a, 3 and 6a in the classification of (3.76)). TC and PF represent transcritical and pitchfork bifurcations respectively.

well as their stability. Then, by the Implicit Function Theorem, small perturbations of higher order terms do not significantly change the local bifurcation curves in the bifurcation diagram. However, some results do not survive and in order to show that, we shall discuss the vector field in cases (I) and (IV) by restoring the remaining normal form coefficients.

$$\begin{aligned} \dot{r} &= \mu_1 r + a_1 r y + a_2 r^3 + a_3 r y^2, \\ \dot{y} &= y(\mu_2 - y - r^2 + b_3 y^2). \end{aligned} \tag{3.88}$$

The equilibria E_0 and E_2 are not affected by the presence of new normal form coefficients. The equilibria E_1 and E_3 also remain there, even though their locations in the phase portrait are slightly affected. The coordinate of E_1 is obtained by solving these equations below,

$$\mu_2 - y - r^2 + b_3 y^2 = 0 \quad \text{and} \quad r = 0. \tag{3.89}$$

Solving the system below gives us the coordinates of E_3,

$$\mu_2 - y - r^2 + b_3 y^2 = 0 \quad \text{and} \quad \mu_1 + a_1 y + a_2 r^2 + a_3 y^2 = 0. \tag{3.90}$$

Thus we obtain the approximate coordinates for both fixed points,

- $E_1 = (r_1, y_1) = (0, \mu_2 + \mathcal{O}(|\mu_2|))$ and
- $E_3 = (r_3, y_3) = (\sqrt{\frac{\mu_1 + a_1\mu_2}{a_1 - a_2}} + \mathcal{O}(|\mu_1 + a_2\mu_2|^2), -\frac{\mu_1 + a_2\mu_2}{a_1 - a_2} + \mathcal{O}(|\mu_1 + a_2\mu_2|)).$

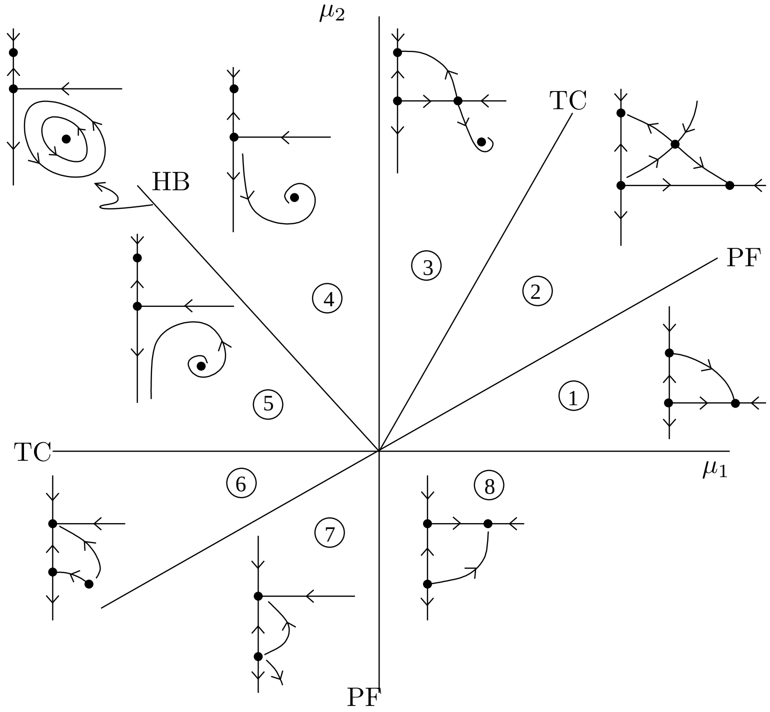

Figure 3.17: Bifurcation diagram of the system (3.84) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (IV) of the unfolding program (or case 2b, 5 and 6b in the classification of (3.76)). TC, PF and HB represent transcritical, pitchfork and Hopf bifurcations respectively.

The system (3.88) may have another equilibrium, that comes from solving (3.90), as they are quadratic equations. We do not worry about this equilibrium since it is located outside any sufficiently small neighbourhood of the origin of the phase plane and does not interact with any of our E_k, $k = 0, \ldots, 3$.

The first transcritical bifurcation occurs when the y-coordinate of the fixed point E_1 goes to zero. Thus, the transcritical bifurcation line $\mu_2 = 0$ is not affected by the addition of new normal form coefficients. This is also the case for the first pitchfork bifurcation $\mu_1 = 0$, at which the equilibrium E_2 branches from E_0. The second pitchfork bifurcation, which is a bifurcation between E_1 and E_3, is slightly affected. We can see this by investigating the fact that this bifurcation occurs when the r-coordinate of the fixed point E_3 goes to zero at $\frac{\mu_1+a_1\mu_2}{a_1-a_2} + \mathcal{O}((\mu_1 + a_2\mu_2)^2)$. Since we are only interested in analysing the phase portrait in the neighbourhood of the origin, the curve of this bifurcation does not qualitatively change. The second transcritical bifurcation, at which the equilibrium E_3 coincides with E_2 and exchanges stability, occurs when the y-coordinate of the fixed point E_3 goes to zero. The location of this bifurcation is $\mu_1 + a_2\mu_2 = 0$. It turns out that new normal form coefficients do not affect the second transcritical bifurcation. Finally, we compute the location of the Hopf bifurcation of (3.88). The trace and the determinant of the Jacobian matrix of system (3.88) evaluated at the fixed point E_3 are respectively,

$$\mathrm{Trace}J(E_3) = 2((a_2 + \frac{1}{2})\mu_1 + 2a_2(a_1 + \frac{1}{2})\mu_2)(a_1 - a_2)^{-1} + \mathcal{O}(|\mu_1 + a_2\mu_2|^2)$$

and

$$\det J(E_3) = -2(\mu_1 + a_1\mu_2)(\mu_1 + a_2\mu_2)(a_1 - a_2)^{-1} + \mathcal{O}(|\mu_1 + a_2\mu_2|^2).$$

We conclude that these normal form coefficients do not significantly affect the local bifurcation curves that are presented in Figures 3.14-3.17. However, the addition of these higher order terms permits us to have global bifurcations such as the birth of an isolated limit cycle, or a heteroclinic orbit, provided that the Hopf bifurcation exists in some of those cases. We present the complete bifurcation diagrams of system (3.88) for cases (I) and (IV) as these cases are the cases that exhibit global bifurcations.

The bifurcation diagram of case (I) is depicted in Figure 3.18. We have a new phase portrait as a result of a global bifurcation curve, which is in fact a heteroclinic bifurcation. If we start from the area above the Hopf bifurcation curve, the corresponding fixed point is unstable. When we cross the Hopf bifurcation the equilibrium is now asymptotically stable, and there appears an unstable isolated limit cycle. Then we go down to cross the heteroclinic bifurcation and the limit cycle collapses as we have a heteroclinic link between the fixed points E_1 and E_2 as illustrated in Figure 3.19. We note that the heteroclinic bifurcation curve takes place below the Hopf bifurcation curve as we assume that the signs of the coefficients are $a_3 < 0$ and $b_3 > 0$ respectively.

The bifurcation diagram for case (IV) can be seen in Figure 3.20. We compare this figure to the bifurcation diagram of the truncated system (3.84) in Figure 3.17. We see that the phase portrait of area 5 of the truncated system is not topologically equivalent with that of the system (3.88). We consider the phase portraits in the area above the Hopf bifurcation in Figure 3.20. As we cross the Hopf bifurcation line, the equilibrium changes stability and

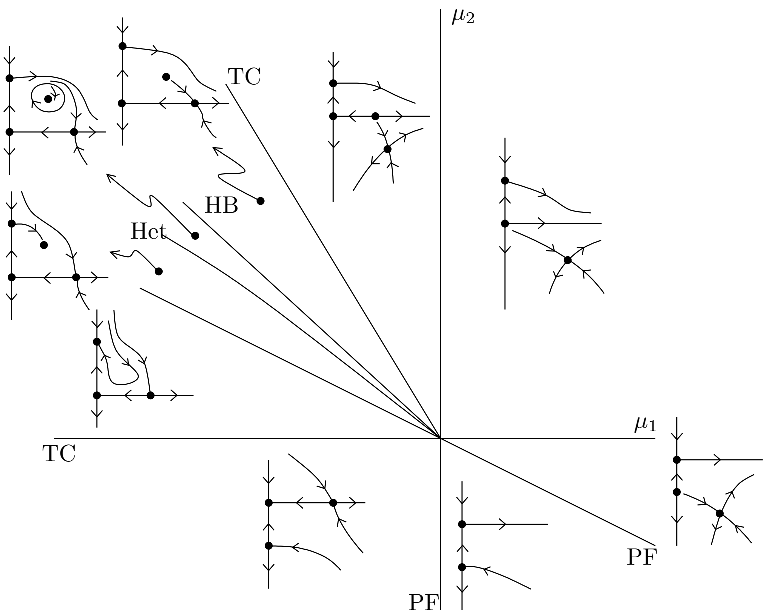

Figure 3.18: Complete bifurcation diagram of the system (3.88) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (I), while a_3 and b_3 are negative and positive respectively. TC, PF and HB represent transcritical, pitchfork and Hopf bifurcations respectively, while Het is a heteroclinic bifurcation.

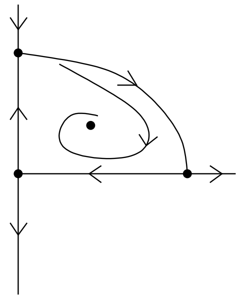

Figure 3.19: The heteroclinic bifurcation occurs when the parameters μ_1 and μ_2 at the heteroclinic bifurcation curve.

a stable isolated limit cycle appears. As we go down to cross the transcritical bifurcation, the period of the limit cycle tends to infinity. The cycle is collapsed and we have no more periodic orbit. All these phenomena are obtained by assuming the signs of the coefficients a_3 and b_3 are respectively $a_3 > 0$ and $b_3 < 0$. As the signs of these coefficients change, the dynamics of the phase portraits near the Hopf bifurcation curve will also change. Thus we have a complete unfolding of a planar system (3.88) in the neighbourhood of a single-zero and a purely imaginary degeneracy ($\mu_1 = \mu_2 = 0$). Up to this point the unfoldings of these four cases of the planar system (3.88) are essentially complete.

Before going to translate the results we have to the three dimensional system (3.67) we shall derive the equation of a global bifurcation of case (I) in Figure 3.18. The analysis now proceeds in a manner parallel to the analysis of the global bifurcation in the previous section of the double-zero degeneracy. We shall rescale the variables r and y along with the unfolding parameters:

$$r = \sqrt{\epsilon}u, \quad y = \epsilon v, \quad \mu_1 = \epsilon\beta_1, \quad \mu_2 = -\epsilon(\frac{1+2a_2}{a_2(1+2a_1)}) + \epsilon^2\beta_2, \tag{3.91}$$

and rescale time $t \mapsto \epsilon t$, so that (3.88) becomes

$$\begin{aligned} \dot{u} &= \beta_1 u + a_1 uv + a_2 u^3 + \epsilon(a_3 uv^2), \\ \dot{v} &= -(\frac{1+2a_2}{a_2(1+2a_1)})\beta_1 v - v^2 - u^2 v + \epsilon(\beta_2 v + b_3 v^3). \end{aligned} \tag{3.92}$$

So now our problem becomes a perturbation of an integrable system:

$$\begin{aligned} \dot{u} &= \beta_1 u + a_1 uv + a_2 u^3, \\ \dot{v} &= -(\frac{1+2a_2}{a_2(1+2a_1)})\beta_1 v - v^2 - u^2 v, \end{aligned} \tag{3.93}$$

with an integral (for $a_1, a_2 \neq 0$ and for $a_1, a_2 \neq -\frac{1}{2}$):

$$F(u,v) = u^{l_1} v^{l_2} (\frac{\beta_1}{l_2} + \frac{a_1}{l_2+1} v + \frac{a_2}{l_2} u^2),$$

where $l_1 = \frac{1+2a_2}{a_1-a_2}$ and $l_2 = \frac{a_2(1+2a_1)}{a_1-a_2}$. We recall that the case of interest which exhibits a heteroclinic bifurcation is case (I) where we have $a_2 > 0$ and $a_1 - a_2 > 0$. Without loss of generality we can set $\beta_1 = -1$ since Hopf bifurcation occurs when the sign of μ_1 is negative, see Figure 3.14. The variation of μ_1 is obtained as ϵ is varied. It is more convenient to work with the system (3.92) multiplied by the integrating factor $u^{l_1-1}v^{l_2-1}$,

$$\begin{aligned} \dot{u} &= \beta_1 u^{l_1}v^{l_2-1} + a_1 u^{l_1}v^{l_2} + a_2 u^{l_1+2}v^{l_2-1} + \epsilon(a_3 u^{l_1}v^{l_2+1}), \\ \dot{v} &= -(\frac{1+2a_2}{a_2(1+2a_1)})\beta_1 u^{l_1-1}v^{l_2} - u^{l_1-1}v^{l_2+1} - u^{l_1+1}v^{l_2} + \\ & \quad \epsilon(\beta_2 u^{l_1-1}v^{l_2} + b_3 u^{l_1-1}v^{l_2+2}). \end{aligned} \tag{3.94}$$

Applying Melnikov theory and Green's theorem, if we have a closed curve Γ for some value β_2 then we have the following equation:

$$\int_{\text{int }\Gamma} \text{trace } D\mathbf{l}(\mathbf{w},\beta_2)\ \mathrm{d}\mathbf{w} = 0, \tag{3.95}$$

where $\mathbf{w} = (u, v)$, $\mathbf{l}(\mathbf{w},\beta_2) = (a_3 u^{l_1}v^{l_2+1}, \beta_2 u^{l_1-1}v^{l_2} + b_3 u^{l_1-1}v^{l_2+2})$. However, in order to locate the global bifurcation taking place in our bifurcation diagram, we shall take an example of case (I). We choose $a_1 = 3/2$ and $a_2 = 1/2$ to get:

$$\text{trace of } D\mathbf{l} = (2a_3 + 4b_3)uv^3 + 2\beta_2 uv.$$

Then we have to integrate:

$$\int\int_{\text{int }\Gamma} \text{trace } D\mathbf{l}(\mathbf{w},\beta_2)du\,dv, \tag{3.96}$$

to find the value of β_2. A closed curve Γ is given by the following equation:

$$0 = F(u,v) = u^2v^2(\frac{\beta_1}{2} + \frac{1}{2}v + \frac{1}{4}u^2).$$

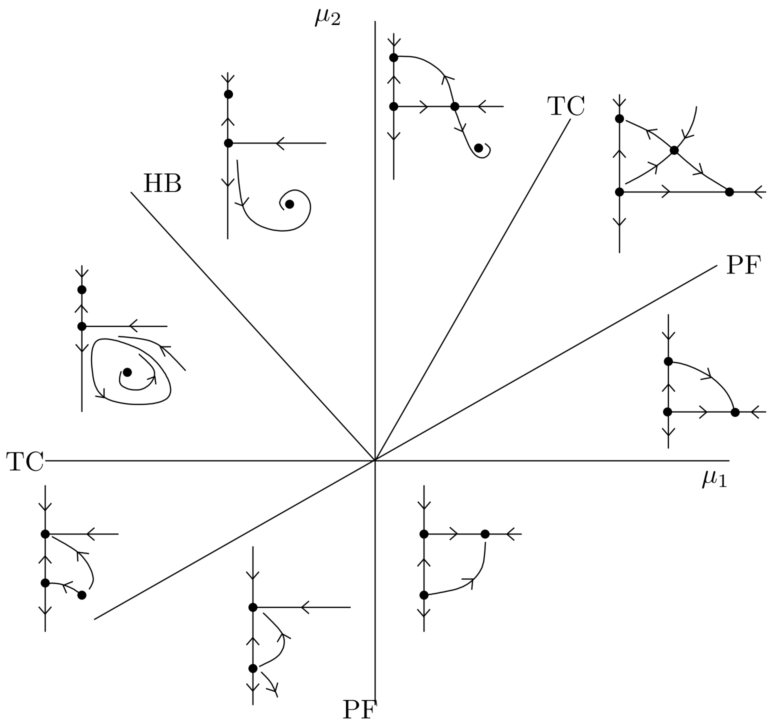

Figure 3.20: Complete bifurcation diagram of the system (3.88) with μ_1 and μ_2 as parameters. The values of a_1 and a_2 satisfy conditions in case (IV), while a_3 and b_3 are positive and negative respectively. TC, PF and HB represent transcritical, pitchfork and Hopf bifurcations respectively..

Evaluating the integral above gives an equation for the global bifurcation up to order ϵ in terms of the parameter β_2,

$$\frac{a_3 + 2b_3}{10} + \frac{\beta_2}{3} = 0.$$

Putting back the parameters used before scaling (3.91), we obtain the location of the global bifurcation curve in our parameter space,

$$\mu_2 = -2\mu_1 - \frac{3}{10}(a_3 + 2b_3)\mu_1^2. \tag{3.97}$$

This bifurcation curve is depicted in Figure 3.18.

3.5.3 Implications in the three dimensional system

In this section, we are going to translate all results previously obtained in the unfolded planar system (3.88) to the unfolded three dimensional system below,

$$\begin{aligned}
\dot{r} &= \mu_1 r + a_1 r y + a_2 r^3 + a_3 r y^2 + \mathcal{O}(\|(r,y)\|^4), \\
\dot{\theta} &= \omega + \mathcal{O}(\|(r,y)\|^2), \\
\dot{y} &= \mu_2 y - y^2 - r^2 y + b_3 y^3 + \mathcal{O}(\|(r,y)\|^4)).
\end{aligned} \tag{3.98}$$

Firstly we consider the truncated system above where higher order terms are not included. We shall analyse what the fixed points in planar system are going to be in the truncated three-dimensional system above. And then we shall translate those bifurcations that occur in the planar system to the truncated three-dimensional system. And lastly we shall consider global dynamics in the planar systems such as the birth of an isolated limit cycle and a heteroclinic bifurcation. However, we may have cases that some of the dynamics disappears once we have perturbations of higher order terms and non-S^1 symmetric terms.

The fixed points that are on the y-axis, E_0 and E_1, correspond to fixed points in the full system, while the fixed points that are not on the $y-$axis which are E_2 and E_3 correspond to limit cycles in the three dimensional space. The stabilities of these fixed points and these limit cycles are the same as those of fixed points in the planar system. Moreover, if these fixed points and limit cycles are hyperbolic, they will persist for small perturbations such as higher order terms, though the equilibrium on the $y-$axis may leave if there is a non-S^1 symmetric perturbation.

The transcritical bifurcation between E_0 and E_1 will become another transcritical bifurcation for system (3.98). The pitchfork bifurcation in which the fixed point E_2 starts to appear now becomes a Hopf bifurcation in the full system. This agrees with the fact that the fixed point E_2 is actually a limit cycle in the full system. Note that the limit cycle E_2 lies inside the invariant manifold $y = 0$. The secondary pitchfork bifurcation, in which the equilibrium E_3 comes into view, is now a Hopf bifurcation. A second transcritical bifurcation, in which the equilibria E_2 and E_3 coincide and exchange their stability, is now a transcritical bifurcation between two periodic solutions. To our best knowledge, this case rarely occurs in the general system. We depict an example of these dynamics translated to the full system in Figure 3.21. We now translate the Hopf bifurcation of fixed point E_3 in the planar system. It turns out that it becomes a Hopf bifurcation of a periodic orbit that is a so-called the

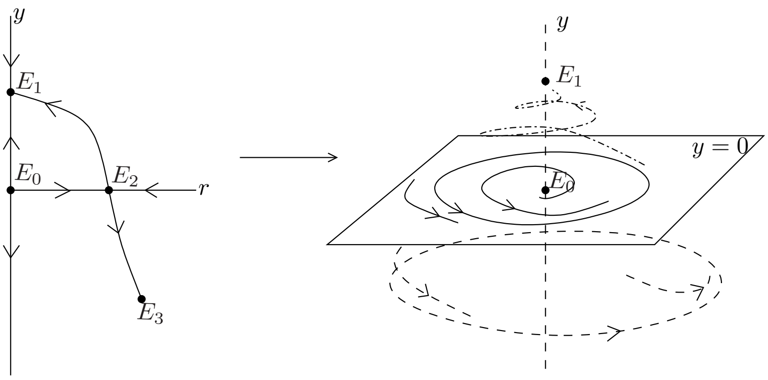

Figure 3.21: Three-dimensional flow with respect to the flow in planar system. We see that E_0 and E_1 stay as fixed points and E_2 and E_3 become periodic solutions.

Neimark-Sacker bifurcation. Furthermore, the closed orbit in the planar system represents an invariant torus in the three-dimensional system. The heteroclinic link that is depicted in Figure 3.19 corresponds to a half sphere in the full system.

Recall that the previous implications for the three-dimensional system concern the truncated system (3.98) where we do not have the perturbation of the higher order terms. The addition of higher order terms does not affect the existence and the stability of the fixed points and the periodic orbits for sufficiently small $||(\mu_1, \mu_2)||$ because of the fact that they are hyperbolic. Then, by using the Implicit Function Theorem for a sufficiently small neighbourhood of $(\mu_1, \mu_2) = (0, 0)$, higher order terms do not affect the local bifurcation curves which are transcritical, Hopf and Neimark-Sacker bifurcations. However, adding higher-order terms will result in topologically non-equivalent bifurcation diagrams as the truncated system has some degenerate features that disappear under perturbations by these higher order terms.

Let us first explain a simple case that is sensitive to the addition of higher order terms. Consider the phase portrait of case (II), depicted in Figure 3.15 in area 2. It has two saddle-type equilibria on the $y-$axis. This axis in fact is invariant due to the S^1 symmetry that connects the one-dimensional stable manifold of one fixed point to the one-dimensional unstable manifold of the other; thus we have a heteroclinic link for all values of μ_1 and μ_2 in this region. The addition of general higher-order terms or in particular, the addition of non-S^1 symmetric terms will make the link disappear. Thus, generically we do not have a heteroclinic link between these two fixed points. We note that this phenomenon does not occur only in case (II) but also in all cases where we have two saddle-type equilibria on the $y-$axis.

The other dynamics that most likely disappears is the global bifurcation phenomenon. Let us consider the heteroclinic orbit in Figure 3.19. We know that in $\mathbb{R}^3$ the heteroclinic link becomes a sphere that is cut in half by the codimension-one invariant manifold possessed by the system. The half sphere is formed by the two-dimensional unstable manifold of the fixed point E_1 and the two-dimensional stable manifold of the fixed point E_2. Thus this half sphere is a result of two surfaces perfectly coinciding. This is an extremely degenerate structure that most likely disappears when higher order terms are added. Generally either we have no intersection at all between these two-dimensional manifolds or, we have a transversal intersection of these manifolds which leads to a transversal heteroclinic orbit in the three dimensional system.

The other phenomenon that previously did not occur and is now possible is the Shilnikov homoclinic bifurcation. Let us consider the fixed point E_1 in Figure 3.19. The addition of higher order terms can destroy the S^1 symmetry, thus the $y-$ axis is no longer invariant. Then the stable manifold of this fixed point, which previously lies inside the $y-$axis, can coincide with the unstable manifold of the same fixed point forming a homoclinic orbit. This bifurcation can possibly lead to exotic dynamics such as chaotic dynamics.

As we discussed earlier, an invariant torus appears through a non-degenerate Neimark-Sacker bifurcation. Under a variation of parameters, a quasi-periodic orbit is born and dies, this is called a phase locking of a periodic orbit. This is another exotic dynamics that can be investigated. To end this discussion we note that we do not prove the existence of these dynamics, we only mention that the dynamics described above can possibly occur.

Remarks

- The single-zero combined with a purely imaginary degeneracy that occurs in the general system gives us the Fold-Hopf bifurcation. The truncated system of this degeneracy is studied in great detail in many bifurcation text books [78]. The reader can also read more information about the implications of the truncated system for the full three dimensional system in these books.

- The blowing-up method that we used in this section was first introduced by Takens [116]. That paper provided the blowing-up for a double-zero, a single-zero combined with a pair of purely imaginary, and two pairs of purely imaginary cases. This method is also explained in great detail in Broer et al. [12] in which polar blowing-up as well as directional blowing-up is discussed. In the discussion in this chapter, we have performed a successive polar and directional blowing-ups.

- In 1969, Shilnikov described a bifurcation involving a homoclinic orbit from a saddle-type equilibrium. Shilnikov proved that the existence of such an orbit, commonly referred to as the Shilnikov homoclinic bifurcation, leads to the existence of infinitely many periodic orbits, i.e. a route to chaos. A good reconstruction of Shilnikov bifurcation from the single-zero combined with a purely imaginary degeneracy is explained in great detail in Wiggins [126]. The Shilnikov bifurcation is also found in an application [120].

- The problem of a quasi-periodic orbit on the surface of an invariant torus of the Fold-Hopf bifurcation has been studied by Scheurle and Marsden [109]. The reconstruction of such a quasi-periodic orbit is explained in Kuznetsov [78, chapter 7].

3.6 Discussion

In this chapter, we showed that the bifurcations of a system with a special structure (i.e. a codimension-one invariant manifold) are different from those of a general dynamical system.

We showed that a codimension-one invariant manifold structure gives rise to many interesting bifurcations, which in particular are one codimension-one bifurcation and three codimension-two bifurcations. This number is less than the number of bifurcations of a general dynamical system, due to the restriction imposed by the special structure. For each bifurcation, the normal form of a system with a codimension-one invariant manifold is derived and treated by the same methods as the normal form of a general dynamical system. Thus, most of the analysis of each bifurcation here is analogous to the analysis of bifurcations of a general dynamical system.

Our results can be applied directly to a system that possesses the same special structure which is a codimension-one invariant manifold. We also require that this manifold is preserved under a variation of parameters. One system that has this special structure is actually discussed in chapter three of this thesis, i.e. the two-dimensional Lotka-Volterra system with a constant term. The result of this chapter explains why the Lotka-Volterra system has two unusual bifurcations in the first place.

We recall that there are two saddle-node–transcritical interactions in the Lotka-Volterra system with a constant term. The first interaction that is a single-zero eigenvalue with higher order term degeneracy is discussed in section three of this chapter. This is also true for the second interaction that has a double-zero eigenvalue degeneracy. This is discussed in section four.

It is true that this chapter has not discussed the Lotka-Volterra system with constant terms. However, if we think about it one more time, we have discussed something that is much more general than the Lotka-Volterra system with constant terms. We have discussed a general system that has a special structure that is actually possessed by the Lotka-Volterra system with constant terms. Thus in this sense, we have done a global analysis in which the Lotka-Volterra system is included.

We remark that the problem in the single-zero and the pair of purely imaginary eigenvalues degeneracy is an interesting topic for future research. In our analysis, the full system is reduced to a planar system featuring many interesting bifurcations. We can translate those bifurcations to the full system which is three-dimensional, which leads to the occurrence of bifurcations that are known to exist in three-dimensional vector fields but are not described by the planar approximation. It will be an interesting challenge to study the existence of such bifurcations.

CHAPTER 4

First integrals of Lotka Volterra systems with constant terms

4.1 Introduction

Consider a general dynamical system in $n-$dimensions as follows

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n \text{ with } n > 1, \tag{4.1}$$

where $\mu \in \mathbb{R}^p$ is a parameter. Our main objective is to find conditions on the parameters such that the system above possesses a first integral. In this chapter, we concentrate on finding integrals of Lotka-Volterra systems with constant terms.

The Lotka-Volterra (LV) system has been the subject of intensive study during the past century. The interaction of two species in an ecosystem [58], a metamorphosis of turbulence in plasma physics [7], hydrodynamic equations, autocatalytic chemical reactions and many more, are of Lotka-Volterra type. Given that there are many applications based on such systems, it is important to understand the dynamics of Lotka-Volterra systems. Nevertheless, the dynamics of such systems is far from being understood. Thus, our main motivation to find first integrals of Lotka-Volterra systems or any dynamical system is the fact that first integrals give global information about the long-term behaviour of such systems.

In two-dimensional systems, the existence of a first integral implies that the system is completely integrable because the phase portraits are completely characterized. For three-dimensional systems, the existence of a first integral means that the system cannot have chaotic motions as the solution will live inside the level sets of such an integral function. The existence of a second integral gives a completely integrable case.

Many different methods have been developed to study the existence of first integrals of Lotka-Volterra systems. Perhaps, one of the earliest attempts to study the existence of first integrals was Cairó et al. [18] who studied the integrability of n-dimensional Lotka-Volterra equations using the Carleman embedding method. They sought a constant of motion (i.e. invariant) that may be either time-dependent or time-independent. There is also Cairó and

Llibre [24] who used a polynomial inverse integrating factor to find a condition for the existence of the first integral. The Darboux method that uses the relationship between algebraic curves and integrability of differential equations has been introduced by Cairó and Llibre [22] to study two-dimensional Lotka-Volterra systems. Cairó et al. (1999) [20] also used the same method to search for a first integral of two-dimensional quadratic systems.

In the three-dimensional case, the Darboux method has also been used to derive an integral for three-dimensional Lotka-Volterra systems [14] and for the so-called ABC systems, which correspond to particular cases of three-dimensional Lotka-Volterra systems where the linear and the diagonal terms are absent. The ABC systems were among the first three-dimensional models that were investigated. One of the first studies was by Grammaticos et al. in 1990 [46] in which the authors derived first integrals using the Frobenius Integrabilty Theorem method (first introduced by Strelcyn and Wojciechowski [115]). Ollagnier [96] has found polynomial first integrals of the ABC system.

Gao and Liu 1998 [39] presented a method that basically relies on changing variables to transform three-dimensional Lotka-Volterra systems to two-dimensional ones. The existence of first integrals follows from integrating the two-dimensional systems. Gao [37] used a direct integration method to find first integrals of three-dimensional Lotka-Volterra systems. A new algorithm presented by Gonzalez-Gascon and Peralta Salas [43] also used three-dimensional Lotka-Volterra systems as an example of their method to find first integrals.

Other intensive research on finding first integrals is using the idea of associating a Hamiltonian to a first integral of a vector field. It was introduced by Nutku [100]. A generalization of this idea to two-dimensional vector fields having a first integral was provided by Cairó and Feix [16], in which through time rescaling the first integral can be considered as a Hamiltonian. Using this relation, Cairo et al. [19] and Hua et al. [63] used an Ansatz for their Hamiltonian functions. They assumed that a first integral (or an invariant) H is a product of two (or three) functions, $H = P(x, y)(Q(x, y))^{\mu}(R(x, y))^{\nu}$. Subsequently, they derived conditions for two-dimensional quadratic systems to have a first integral.

Another Hamiltonian method that has been used is as follows. A general system (4.1) is said to have a Hamiltonian structure if and only if it can be written as $\dot{x} = f(x) = S(x)\nabla H(x)$, where S is a skew-symmetric matrix and H is a smooth function. The matrix function S must satisfy the Jacobi identity [38]. Plank [102] has used this property to find a Hamiltonian function for two-dimensional Lotka-Volterra systems, while Gao [38], using the same property, has derived conditions for three-dimensional systems not to be chaotic.

However in this chapter, we are not going to use the Jacobi identity property for the matrix S since what we only need is the fact that H is a first integral, thus the fact that $f(x)$ can be written as $S(x)\nabla H(x)$ is enough for us now. Consider the following proposition.

Proposition 4.10 (McLachlan et al. 1999 [94]). *Let $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, $n > 1$, be a vector field and $H \in C(\mathbb{R}^n, \mathbb{R})$ is a first integral of the vector field f (i.e. $f.\nabla H = 0$) for all x. Then there is a skew-symmetric matrix function $S(x)$ on the domain $\{x : \nabla H \neq 0\}$ such that $f = S\nabla H$.*

As a consequence of the proposition above, there is also a skew symmetric matrix function

$T(x)$ on the domain $\{x : f \neq 0\}$ such that $\nabla H = Tf$. We are going to use this idea to find first integrals and their constraints for two- and three-dimensional Lotka-Volterra systems with constant terms. We call the matrix T an integrating factor matrix. As the function H is a first integral, we must have $\mathrm{curl}(\nabla H) = 0$. This implies that

$$\mathrm{curl}(\,Tf\,) = 0. \tag{4.2}$$

The above condition will be a condition for a matrix T to be an integrating factor. Making an Ansatz concerning the integrating factor matrix T, we obtain both integrals as well as conditions on the parameters for the existence of the integrals. However, the meaning of curl will be different in every dimension greater than one. For two dimensional systems, $\mathrm{curl}(Tf)$ will be just a scalar as follows,

$$\frac{\partial (Tf)_1}{\partial x_2} - \frac{\partial (Tf)_2}{\partial x_1} = 0,$$

where $(TF)_i$ is the $i-th$ component of the vector Tf. In three-dimensional systems, $\mathrm{curl}(Tf)$ is a vector in $\mathbb{R}^3$, as a result of a cross product of the vectors ∇ and Tf.

We shall use the integrating factor matrix to find first integrals and their conditions. Integrability of two-dimensional Lotka-Volterra systems with constant terms will be investigated in the next section. Conditions and integrals are derived using an Ansatz on the integrating factor matrix. First integrals of three-dimensional Lotka-Volterra systems are discussed in section three. Finally, some remarks comparing known methods and our method to find first integrals, along with a conclusion and suggestions for future research, are discussed in section four. Note that we have not (yet) recovered all known integrals of LV systems with constant terms, so there is still plenty to be done.

4.2 Two-dimensional LV systems with constant terms

In this section, we consider integrals of the two-dimensional Lotka-Volterra system with constant terms:

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2) = x_1(b_1 + a_{11}x_1 + a_{12}x_2) + e_1, \\ \dot{x_2} &= f_2(x_1, x_2) = x_2(b_2 + a_{21}x_1 + a_{22}x_2) + e_2, \end{aligned} \tag{4.3}$$

where b_i, a_{ij} $(i, j = 1, 2)$ are arbitrary parameters and e_1, e_2 are the constant terms. We choose an integrating factor matrix as follows,

$$T(x_1, x_2) = \begin{pmatrix} 0 & -\alpha R \\ \alpha R & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \tag{4.4}$$

where we make the Ansatz that $R = R(x_1, x_2) = x_1^{l_1 - 1} x_2^{l_2 - 1}$, l_1 and l_2 are free parameters that are to be determined later on. The matrix $T(x_1, x_2)$ is an integrating factor if and only if the curl of Tf is zero, where $f = (f_1, f_2)$. As is mentioned before, in the two dimensional case, this condition is equivalent to the following,

$$\frac{\partial (\alpha R f_1)}{\partial x_1} + \frac{\partial (\alpha R f_2)}{\partial x_2} = 0. \tag{4.5}$$

The associated first integral H is given by

$$H(x_1, x_2) = \int R(x_1, x_2) f_1(x_1, x_2)\, dx_2 + h(x_1), \tag{4.6}$$

where $h(x_1)$ is found by imposing $\partial H/\partial x_1 = -Rf_2$.

We then multiply the matrix (4.4) and the right-hand side of the vector field (4.3) and substitute them in the equation (4.5) to get,

$$0 = x_1^{l_1-1} x_2^{l_2-1}[b_1 l_1 + b_2 l_2 + (l_1 a_{11} + l_2 a_{21} + a_{11})x_1 + \tag{4.7}$$

$$(l_1 a_{12} + l_2 a_{22} + a_{22})x_2 + (l_1 - 1)e/x_1 + (l_2 - 1)f/x_2].$$

The expression above is satisfied if and only if the parameters satisfy these five conditions:

$$e_1 l_1 = e_1, \tag{4.8}$$

$$e_2 l_2 = e_2, \tag{4.9}$$

$$l_1 a_{11} + l_2 a_{21} = -a_{11}, \tag{4.10}$$

$$l_1 a_{12} + l_2 a_{22} = -a_{22}, \tag{4.11}$$

$$b_1 l_1 + b_2 l_2 = 0. \tag{4.12}$$

We can write (4.8-4.12) as an overdetermined linear system:

$$Al = r, \tag{4.13}$$

where $l = (l_1, l_2)$ and the matrix A and the vector r are to be determined later. The system has a solution only if the vector r is orthogonal to the left null space of the matrix A. In the following we give the corresponding equations along with the resulting integrals for the cases $e_1, e_2 \neq 0$, $e_1 \neq 0, e_2 = 0$ and $e_1 = e_2 = 0$ separately. The case $e_1 = 0, e_2 \neq 0$ follows by symmetry considerations. We note that the final case, which corresponds to the original Lotka-Volterra system where $e_1 = e_2 = 0$, has been discussed by various people. However we shall also discuss this case in order for the presentation to be self-contained.

4.2.1 The case $e_1, e_2 \neq 0$

We have the case where both the constant terms e_1 and e_2 are non-zero. Then by (4.8) and (4.9), this implies that $l_1 = l_2 = 1$. Moreover, we can simplify the other conditions to:

$$b_1 + b_2 = 0, \quad 2a_{11} + a_{21} = 0, \quad \text{and} \quad a_{12} + 2a_{22} = 0. \tag{4.14}$$

If the Lotka-Volterra system (4.3) with non-zero constant terms e and f satisfies conditions (4.14), then it has a first integral that is given by:

$$H = b_1 x_1 x_2 + a_{11} {x_1}^2 x_2 - a_{22} x_1 {x_2}^2 + e_1 x_2 - e_2 x_1. \tag{4.15}$$

4.2.2 The case $e_1 \neq 0$, $e_2 = 0$

One of the constant terms, e_1 is not zero. This means that the free parameter l_1 must be 1 by (4.8). We now have a linear system like (4.13) with a 3 by 1 matrix A and a vector r in $\mathbb{R}^3$ with only one unknown l_2. Without loss of generality, we assume that the matrix A is of rank 1, when A has rank zero we have a trivial integral $H = x_2$ since $\dot{x}_2 = 0$. Using the fact that this system must be solvable, we can again find the conditions for the existence of the first integral. First we assume that $a_{21} \neq 0$. In that case the solvability conditions are given by:

$$\frac{2a_{11}a_{22}}{a_{21}} - a_{22} - a_{12} = 0, \quad \text{and} \quad 2b_2a_{11} - b_1a_{21} = 0. \tag{4.16}$$

If we have $a_{21} = 0$ and $a_{22} \neq 0$ then we can compute the solvability conditions:

$$\frac{b_2}{a_{22}}(a_{22} + a_{12}) - b_1 = 0, \quad \text{and} \quad a_{11} = 0. \tag{4.17}$$

Finally, when $a_{21} = a_{22} = 0$ but $b_2 \neq 0$ the solvability conditions are given by:

$$a_{22} + a_{12} = 0, \quad \text{and} \quad a_{11} = 0. \tag{4.18}$$

Hence when our system satisfies one of the conditions above, there exists a unique solution, l_2. If l_2 is not zero then the first integral is given by:

$$H = {x_2}^{l_2}(-b_2x_1 - \frac{a_{21}}{2}{x_1}^2 - a_{22}x_1x_2 + \frac{e_1}{l_2}). \tag{4.19}$$

However, in the case where the exponent l_2 is zero, the integral is given by:

$$H = -b_2x_1 - \frac{a_{21}}{2}{x_1}^2 - a_{22}x_1x_2 + e_1 \ln|x_2|. \tag{4.20}$$

4.2.3 The case $e_1 = 0$ and $e_2 = 0$

Finally, if $e_1 = e_2 = 0$ equations (4.8) and (4.9) are trivial and we have a linear system of the form (4.13) with a 3 by 2 matrix A and a vector e in $\mathbb{R}^3$ from (4.10-4.12).

If A is of maximal rank then the solvability condition of the linear system (4.13) is given by :

$$b_1a_{22}(a_{21} - a_{11}) + b_2a_{11}(a_{12} - a_{22}) = 0. \tag{4.21}$$

So if our system satisfies the condition above, then there exist l_1 and l_2. When neither l_1 and l_2 are zero, the first integral is given by:

$$H = {x_1}^{l_1}{x_2}^{l_2}(\frac{b_1}{l_2} + \frac{a_{11}}{l_2}x_1 - \frac{a_{22}}{l_1}x_2). \tag{4.22}$$

However, either l_1 or l_2 may be zero and if $l_1 = 0$, $l_2 \neq 0$ and $l_2 \neq -1$ then we have $b_2 = a_{22} = 0$, $a_{11} \neq a_{21}$ and an integral that is given by:

$$H = {x_2}^{l_2}(\frac{b_1}{l_2} + \frac{a_{11}}{l_2}x_1 + \frac{a_{12}}{(l_2+1)}x_2). \tag{4.23}$$

But when $l_1 = 0$ and $l_2 = -1$, this implies $b_2 = 0$ and $a_{21} = a_{11}$. It follows that the integral is given by:

$$H = a_{12}\ln|x_2| - a_{22}\ln|x_1| - \frac{b_1}{x_2} - a_{11}\frac{x_1}{x_2}. \tag{4.24}$$

Finally, when both l_1 and l_2 are zero, which implies $a_{11} = a_{22} = 0$, the first integral is given by:

$$H = b_1 \ln|x_2| + a_{12}x_2 - b_2 \ln|x_1| - a_{21}x_1. \tag{4.25}$$

We remark that the case when $l_2 = 0$, $l_1 \neq 0$, $l_1 \neq -1$ and the case when $l_2 = 0$, $l_1 = -1$ follow by symmetry considerations.

If matrix A has rank 1, we can assume without loss of generality that one of its column vectors, $(a_{11}, a_{12}, b_1)^T$ does not have zero norm and the other is proportional to it, namely:

$$(a_{21}, a_{22}, b_2)^T = \lambda(a_{11}, a_{12}, b_1)^T, \tag{4.26}$$

where $\lambda \in \mathbb{R}$. The solvability conditions depend very much on this vector. We summarize in the following:

- if $a_{11} \neq 0$, then the solvability conditions are $b_1 = 0$ and $a_{12} = a_{22}$. The integral is given by

$$H = x_2 {x_1}^{-1}. \tag{4.27}$$

- If $a_{11} = 0$ and $a_{12} \neq 0$, then the solvability condition is $b_1 = 0$ with the integral:

$$H = x_2 {x_1}^{-\lambda}, \tag{4.28}$$

  where $\lambda = a_{22}/a_{12}$.

- if $a_{11} = 0$, $a_{12} \neq 0$ and $b_1 \neq 0$, then we need $a_{22} = 0$. The integral for this case is of the form,

$$H = x_2. \tag{4.29}$$

- Finally if $a_{11} = a_{12} = 0$ but $b_1 \neq 0$ (since rank of A is one) then the system is already solvable with the following integral:

$$H = x_2 {x_1}^{-\lambda}, \tag{4.30}$$

  where $\lambda = b_2/b_1$.

4.2.4 Further notes regarding the known integrals of two-dimensional LV systems and quadratic systems

As we said earlier, many attempts have been made to study the integrability of two-dimensional Lotka-Volterra systems and quadratic systems. Many people found different first integrals using different methods, and it may be possible that the first integral they found is similar to the ones we have found in this discussion.

The first integral (4.15) along with its conditions (4.14) has been found before. It has been found for the first time perhaps by Frommer (1934) (see Artés and Llibre (1994) [5]). Cairó et al. [19] used the Hamiltonian method to derive first integrals for two-dimensional quadratic systems and one of their results corresponds to the first integral (4.15). Hua et al. [64] studied the connection between the existence of a first integral and the Painlevé property in a general quadratic system. They listed first integrals of quadratic systems that have been found before and (4.15) is one of them. The form of the first integral and the vector field they studied are different, but through some invertible transformations it is not hard to check that they are actually equivalent.

For the first integral in the case $e_1 = 0$ and $e_2 = 0$, it is not possible to list all the first integrals that have been already derived since there are so many. First integrals (4.22-4.25) have been derived by, for instance Nutku (1989), Cairó and Feix [15], Plank [102], Cairó and Llibre (1999) [20] and (2000) [24] using various methods. We remark that the first integral (4.25) that has constrains $a_{11} = a_{22} = 0$ was firstly derived by Volterra himself as a constant of motion, (see the book by Hofbauer and Sigmund (1998) [58]). The other first integrals to our best knowledge, for the cases where $e_1 \neq 0, e_2 = 0$ and $e_1 = e_2 = 0$, seem to be new.

4.3 Three-dimensional LV systems with constant terms

We consider the following three-dimensional Lotka-Volterra systems with constant terms:

$$\begin{aligned}
\dot{x_1} &= f_1(x_1, x_2, x_3) = x_1(b_1 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3) + e_1, \\
\dot{x_2} &= f_2(x_1, x_2, x_3) = x_2(b_2 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3) + e_2, \\
\dot{x_3} &= f_3(x_1, x_2, x_3) = x_3(b_3 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3) + e_3,
\end{aligned} \tag{4.31}$$

where b_i, a_{ij} $(i, j = 1, 2)$ are arbitrary parameters and e_i $(i = 1, 2, 3)$ are the constant terms. In this section, in order to find integrals of the system above we shall make the following two Ansatzs for the skew-symmetric matrix T,

$$T_1(x_1, x_2, x_3) = R \begin{pmatrix} 0 & -\alpha' & -\beta' \\ \alpha' & 0 & -\gamma' \\ \beta' & \gamma' & 0 \end{pmatrix}, \tag{4.32}$$

respectively,

$$T_2(x_1, x_2, x_3) = R \begin{pmatrix} 0 & -\alpha x_3 & -\beta x_2 \\ \alpha x_3 & 0 & -\gamma x_1 \\ \beta x_2 & \gamma x_1 & 0 \end{pmatrix}, \tag{4.33}$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{R}$ are arbitrary parameters. The function R that we use here has the same form that we have used in the two-dimensional case which is $R = R(x_1, x_2, x_3) = x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}$, l_i $(i = 1, 2, 3)$ are free parameters that are to be determined later on. The matrices $T_i(x_1, x_2, x_3)$ $(i = 1, 2)$ are integrating factors if and only if the curls of $T_i f$ are zero, where $f = (f_1, f_2, f_3)$. In the three-dimensional case, this condition is equivalent to

$$\begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial H_i/\partial x_1 & \partial H_i/\partial x_2 & \partial H_i/\partial x_3 \end{vmatrix} = 0, \tag{4.34}$$

where

$$\nabla H_i = T_i f \quad (i = 1, 2). \tag{4.35}$$

We expand the above expression with respect to the matrices T_1 and T_2 as follows,

$$\frac{\partial H_1}{\partial x_1} = -R\alpha' f_2 - R\beta' f_3, \tag{4.36}$$
$$\frac{\partial H_1}{\partial x_2} = R\alpha' f_1 - R\gamma' f_3, \tag{4.37}$$
$$\frac{\partial H_1}{\partial x_3} = R\beta' f_1 + R\gamma' f_2. \tag{4.38}$$

respectively

$$\frac{\partial H_2}{\partial x_1} = -R\alpha x_3 f_2 - R\beta x_2 f_3, \tag{4.39}$$
$$\frac{\partial H_2}{\partial x_2} = R\alpha x_3 f_1 - R\gamma x_1 f_3, \tag{4.40}$$
$$\frac{\partial H_2}{\partial x_3} = R\beta x_2 f_1 + R\gamma x_1 f_2. \tag{4.41}$$

The associated first integral $H_1(x_1, x_2, x_3)$, which corresponds to the matrix T_1, is given by

$$H_1(x_1, x_2, x_3) = \int (R\beta' f_1 + R\gamma' f_2)\, dx_3 + h(x_1, x_2), \tag{4.42}$$

where $h(x_1, x_2)$ is found by imposing (4.36) and (4.37). Respectively, the associated first integral in the case where we use the integrating factor matrix T_2 can be found in a similar way.

In the following, we shall derive integrals for the cases $e_1, e_2, e_3 \neq 0$; $e_1, e_2 \neq 0, e_3 = 0$; $e_1 \neq 0, e_2 = e_3 = 0$; and $e_1 = e_2 = e_3 = 0$.

4.3.1 The case $e_1, e_2, e_3 \neq 0$

We first start discussing the case where all the constant terms, e_1, e_2, e_3 are not zero. We start with the following lemma, which implies that the integrating factor matrix T_2 does not work in this case.

Lemma 4.11. *If $e_1e_2e_3 \neq 0$ then we must have $T_2 = 0$ in order to satisfy condition (4.34) for the case that the subscript $i = 2$.*

Proof. We substitute the expression (4.39-4.41) in the condition (4.34) for the case that the subscript $i = 2$. Therefore, we have the following,

$$\begin{pmatrix} \frac{\partial}{\partial x_2}(R\beta x_2 f_1 + R\gamma x_1 f_2) - \frac{\partial}{\partial x_3}(R\alpha x_3 f_1 - R\gamma x_1 f_3) \\ \frac{\partial}{\partial x_3}(-R\alpha x_3 f_2 - R\beta x_2 f_3) - \frac{\partial}{\partial x_1}(R\beta x_2 f_1 + R\gamma x_1 f_2) \\ \frac{\partial}{\partial x_1}(R\alpha x_3 f_1 - R\gamma x_1 f_3) - \frac{\partial}{\partial x_2}(-R\alpha x_3 f_2 - R\beta x_2 f_3) \end{pmatrix} = 0. \tag{4.43}$$

We then substitute the differential equation (4.31) in the vector above. The first entry, which has to be equal to zero, is equivalent to

$$\begin{aligned} 0 = x_1^{l_1} x_2^{l_2-1} x_3^{l_3-1} [&\beta(l_2 b_1 + l_2 a_{11} x_1 + (l_2+1) a_{12} x_2 + l_2 a_{13} x_3 + l_2 e_1/x_1) \\ &+ \gamma(l_2 b_2 + l_2 a_{21} x_1 + (l_2+1) a_{22} x_2 + l_2 a_{23} x_3 + (l_2-1) e_2/x_2) \\ &- \alpha(l_3 b_1 + l_3 a_{11} x_1 + l_3 a_{12} x_2 + (l_3+1) a_{13} x_3 + l_3 e_1/x_1) \\ &+ \gamma(l_3 b_3 + l_3 a_{31} x_1 + l_3 a_{32} x_2 + (l_3+1) a_{33} x_3 + (l_3-1) e_3/x_3)]. \end{aligned} \tag{4.44}$$

The fact that the second entry of the vector (4.43) is equal to zero, gives us the following equation,

$$
\begin{aligned}
0 = {} & -x_1^{l_1-1}x_2^{l_2}x_3^{l_3-1}[\,\alpha(l_3b_2 + l_3a_{21}x_1 + l_3a_{22}x_2 + (l_3+1)a_{23}x_3 + l_3e_2/x_2) \\
& + \beta(l_3b_3 + l_3a_{31}x_1 + l_3a_{32}x_2 + (l_3+1)a_{33}x_3 + (l_3-1)e_3/x_3) \\
& + \beta(l_1b_1 + (l_1+1)a_{11}x_1 + l_1a_{12}x_2 + l_1a_{13}x_3 + (l_1-1)e_1/x_1) \\
& + \gamma(l_1b_2 + (l_1+1)a_{21}x_1 + l_1a_{22}x_2 + l_1a_{23}x_3 + l_1e_2/x_2)]. \qquad (4.45)
\end{aligned}
$$

Finally, the fact that the third entry of the vector (4.43) is equal to zero, gives us the following equation,

$$
\begin{aligned}
0 = {} & x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3}[\,\alpha(l_1b_1 + (l_1+1)a_{11}x_1 + l_1a_{12}x_2 + l_1a_{13}x_3 + (l_1-1)e_1/x_1) \\
& - \gamma(l_1b_3 + (l_1+1)a_{31}x_1 + l_1a_{32}x_2 + l_1a_{33}x_3 + l_1e_3/x_3) \\
& + \alpha(l_2b_2 + l_2a_{21}x_1 + (l_2+1)a_{22}x_2 + l_2a_{23}x_3 + (l_2-1)e_2/x_2) \\
& + \beta(l_2b_3 + l_2a_{31}x_1 + (l_2+1)a_{32}x_2 + l_2a_{33}x_3 + l_2e_3/x_3). \qquad (4.46)
\end{aligned}
$$

We now want to find conditions on the parameters $(\alpha, \beta, \gamma, l_i, a_{ij}, b_i, e_i)$ such that the above conditions are satisfied. If $e_1e_2e_3 \neq 0$, then it immediately follows that $l_1 = l_2 = l_3 = 1$. Also, from coefficients of $1/x_1$, $1/x_2$, and $1/x_3$ of (4.44-4.46) respectively, we have the following,

$$
\begin{aligned}
\beta e_1 l_2 - \alpha e_1 l_3 &= 0, \\
-\alpha e_2 l_3 - \gamma e_2 l_1 &= 0, \\
-\gamma e_3 l_1 + \beta e_3 l_2 &= 0.
\end{aligned}
$$

As $e_1e_2e_3 \neq 0$ and $l_1 = l_2 = l_3 = 1$, we conclude that

$$
\begin{aligned}
\beta - \alpha &= 0, \\
\alpha + \gamma &= 0, \\
-\gamma + \beta &= 0,
\end{aligned}
$$

which implies that $\alpha = \beta = \gamma = 0$. Thus T_2 has to be a zero matrix. $\square$

Therefore, we shall use the integrating factor matrix T_1 to derive integrals for the case $e_1e_2e_3 \neq 0$. We substitute the expressions (4.36-4.38) in the condition (4.34) for $i = 1$ and we conclude that the following vector must be equal to zero.

$$
\begin{pmatrix}
\dfrac{\partial}{\partial y}(R\beta' f_1 + R\gamma' f_2) - \dfrac{\partial}{\partial z}(R\alpha' f_1 - R\gamma' f_3) \\
\dfrac{\partial}{\partial z}(-R\alpha' f_2 - R\beta' f_3) - \dfrac{\partial}{\partial x}(R\beta' f_1 + R\gamma' f_2) \\
\dfrac{\partial}{\partial x}(R\alpha' f_1 - R\gamma' f_3) - \dfrac{\partial}{\partial y}(-R\alpha' f_2 - R\beta' f_3)
\end{pmatrix} = 0. \qquad (4.47)
$$

We substitute the vector field (4.31) into the above vector. The equation in the following is due to the first entry of the above vector being equal to zero.

$$
x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}\left(\frac{\beta' x_1}{x_2}[(l_2-1)b_1 + (l_2-1)a_{11}x_1 + l_2a_{12}x_2 + (l_2-1)a_{13}x_3\right.
$$

$$+(l_2-1)\frac{e_1}{x_1}]+\gamma'[l_2b_2+l_2a_{21}x_1+(l_2+1)a_{22}x_2+l_2a_{23}x_3+(l_2-1)\frac{e_2}{x_2}]$$
$$-\frac{\alpha' x_1}{x_3}[(l_3-1)b_1+(l_3-1)a_{11}x_1+(l_3-1)a_{12}x_2+l_3a_{13}x_3+(l_3-1)\frac{e_1}{x_1}]$$
$$+\gamma'[l_3b_3+l_3a_{31}x_1+l_3a_{32}x_2+(l_3+1)a_{33}x_3+(l_3-1)\frac{e_3}{x_3}]\Big)=0. \tag{4.48}$$

Substituting f_1, f_2 and f_3 of the vector field (4.31) to the second entry of the vector (4.47), we get the following,

$$-x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}\Big(\frac{\alpha' x_2}{x_3}[(l_3-1)b_2+(l_3-1)a_{21}x_1+(l_3-1)a_{22}x_2+l_3a_{23}x_3$$
$$+(l_3-1)\frac{e_2}{x_2}]+\beta'[l_3b_3+l_3a_{31}x_1+l_3a_{32}x_2+(l_3+1)a_{33}x_3+(l_3-1)\frac{e_3}{x_3}]$$
$$+\frac{\gamma' x_2}{x_1}[(l_1-1)b_2+l_1a_{21}x_1+(l_1-1)a_{22}x_2+(l_1-1)a_{23}x_3+(l_1-1)\frac{e_2}{x_2}]$$
$$+\beta'[l_1b_1+(l_1+1)a_{11}x_1+l_1a_{12}x_2+l_1a_{13}x_3+(l_1-1)\frac{e_1}{x_1}]\Big)=0. \tag{4.49}$$

Finally, the fact that the third entry of the vector (4.47) is equal to zero gives,

$$x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}\Big(-\frac{\gamma' x_3}{x_1}[(l_1-1)b_3+l_1a_{31}x_1+(l_1-1)a_{32}x_2+(l_1-1)a_{33}x_3$$
$$+(l_1-1)\frac{e_3}{x_3}]+\alpha'[l_1b_1+(l_1+1)a_{11}x_1+l_1a_{12}x_2+l_1a_{13}x_3+(l_1-1)\frac{e_1}{x_1}]$$
$$+\frac{\beta' x_3}{x_2}[(l_2-1)b_3+(l_2-1)a_{31}x_1+l_2a_{32}x_2+(l_2-1)a_{33}x_3+(l_2-1)\frac{e_3}{x_3}]$$
$$+\alpha'[l_2b_2+l_2a_{21}x_1+(l_2+1)a_{22}x_2+l_2a_{23}x_3+(l_2-1)\frac{e_2}{x_2}]\Big)=0. \tag{4.50}$$

We now want to find conditions on the parameters $(\alpha', \beta', \gamma', l_i, a_{ij}, b_i, e_i)$ such that the above equations are satisfied. The results for this case are summarized in the following lemma.

Lemma 4.12. *The vector field (4.31) with $e_1, e_2, e_3 \neq 0$ has a first integral in the following cases:*

1. *if the conditions $b_1+b_2=0$, $2a_{11}+a_{21}=0$, $2a_{22}+a_{12}=0$, $a_{13}=a_{23}=0$ are satisfied, then the integral is given by*

$$H = b_1x_1x_2 + a_{11}x_1^2x_2 - a_{22}x_1x_2^2 + e_1x_2 - e_2x_1, \tag{4.51}$$

2. *if the conditions $b_1+b_2=0$, $b_1+b_3=0$, $2a_{11}+a_{21}=0$, $2a_{11}+a_{31}=0$, $2a_{22}+a_{12}=0$, $2a_{33}+a_{13}=0$, and $a_{12}a_{13}+a_{12}a_{23}+a_{13}a_{32}=0$ are satisfied then the integral is given by*

$$\begin{aligned}H = &-2b_1a_{33}x_1x_3 - 2b_1a_{22}x_1x_2 - 2a_{11}a_{33}x_1^2x_3 + 2a_{11}a_{22}x_1^2x_2\\ &+2a_{33}^2x_1x_3^2 + 2a_{22}^2x_1x_2^2 + 4a_{22}a_{33}x_1x_2x_3\\ &+2(e_3a_{33}+e_2a_{22})x_1 - 2e_1a_{33}x_3 + 2e_1a_{22}x_2,\end{aligned} \tag{4.52}$$

3. *if the conditions $b_i = 0$ and $a_{ij} = -2a_{jj}$ for $i \neq j$ and $i, j = 1, 2, 3$ are satisfied, then the integral is given by*

$$\begin{aligned} H = {} & a_{11}^2 a_{22} x_1^2 x_2 - a_{11}^2 a_{33} x_1^2 x_3 - a_{11} a_{22}^2 x_1 x_2^2 + a_{11} a_{33}^2 x_1 x_3^2 + a_{22}^2 a_{33} x_2^2 x_3 \\ & - a_{22} a_{33}^2 x_2 x_3^2 + (-a_{11} a_{22} e_2 + a_{11} a_{33} e_3) x_1 + (a_{11} a_{22} e_1 - a_{22} a_{33} e_3) x_2 \\ & + (a_{22} a_{33} e_2 - a_{11} a_{33} e_1) x_3. \end{aligned} \tag{4.53}$$

Proof. It immediately follows from (4.48-4.50) that l_1, l_2, l_3 must all equal to one. Consequently, we get the following conditions on the parameters,

$$\alpha'(b_1 + b_2) = 0, \tag{4.54}$$
$$\beta'(b_1 + b_3) = 0, \tag{4.55}$$
$$\gamma'(b_2 + b_3) = 0, \tag{4.56}$$
$$\alpha'(2a_{11} + a_{21}) = 0, \tag{4.57}$$
$$\alpha'(2a_{22} + a_{12}) = 0, \tag{4.58}$$
$$\beta'(2a_{11} + a_{31}) = 0, \tag{4.59}$$
$$\beta'(2a_{33} + a_{13}) = 0, \tag{4.60}$$
$$\gamma'(2a_{22} + a_{32}) = 0, \tag{4.61}$$
$$\gamma'(2a_{33} + a_{23}) = 0, \tag{4.62}$$
$$-\alpha' a_{13} + \beta' a_{12} + \gamma'(a_{21} + a_{31}) = 0, \tag{4.63}$$
$$\alpha' a_{23} + \beta'(a_{12} + a_{32}) + \gamma' a_{21} = 0, \tag{4.64}$$
$$\alpha'(a_{13} + a_{23}) + \beta' a_{32} - \gamma' a_{31} = 0. \tag{4.65}$$

1. We start with the case where $\alpha' \neq 0, \beta' = \gamma' = 0$. From (4.54-4.65) the following conditions immediately apply:

$$b_1 + b_2 = 0, 2a_{11} + a_{21} = 0, \tag{4.66}$$
$$2a_{22} + a_{12} = 0, a_{13} = a_{23} = 0, \tag{4.67}$$

and using (4.42) we obtain the first integral (4.51).

2. We turn to the case where $\alpha', \beta' \neq 0, \gamma' = 0$. From (4.54-4.62), the following conditions immediately apply,

$$b_1 + b_2 = 0, 2a_{11} + a_{21} = 0, 2a_{11} + a_{31} = 0, \tag{4.68}$$
$$b_1 + b_3 = 0, 2a_{22} + a_{12} = 0, 2a_{33} + a_{13} = 0. \tag{4.69}$$

While, α' and β' can be computed from the following linear homogeneous equations due to (4.63-4.65),

$$\alpha' a_{13} - \beta' a_{12} = 0, \tag{4.70}$$
$$\alpha' a_{23} + \beta'(a_{12} + a_{32}) = 0, \tag{4.71}$$
$$\alpha'(a_{13} + a_{23}) + \beta' a_{32} = 0. \tag{4.72}$$

The solvability condition of the linear system above is given by,

$$a_{12}a_{13} + a_{12}a_{23} + a_{13}a_{32} = 0. \tag{4.73}$$

If the above condition is satisfied, we obtain the first integral (4.52) due to (4.42).

3. Finally we consider the case $\alpha', \beta', \gamma' \neq 0$. From the equations (4.54-4.62), we have

$$b_1 + b_2 = 0, 2a_{11} + a_{21} = 0, 2a_{22} + a_{12} = 0, \tag{4.74}$$

$$b_1 + b_3 = 0, 2a_{11} + a_{31} = 0, 2a_{33} + a_{13} = 0, \tag{4.75}$$

$$b_2 + b_3 = 0, 2a_{22} + a_{32} = 0, 2a_{33} + a_{23} = 0. \tag{4.76}$$

We simplify the equations above to

$$b_i = 0 \quad \text{and} \quad a_{ij} = -2a_{jj}, \quad i \neq j, \ (i, j = 1, 2, 3). \tag{4.77}$$

The parameters α', β' and γ' can be computed from the equations (4.63-4.65), giving us the following homogenous linear system,

$$\begin{pmatrix} -a_{13} & a_{12} & (a_{21} + a_{31}) \\ a_{23} & (a_{12} + a_{32}) & a_{21} \\ (a_{13} + a_{23}) & a_{32} & -a_{31} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0. \tag{4.78}$$

Substituting (4.77) to the above linear system, we have the following solutions,

$$\alpha = a_{11}a_{22}\mu, \quad \beta = -a_{11}a_{33}\mu, \quad \gamma = a_{22}a_{33}\mu. \tag{4.79}$$

Then if all the parameters of the Lotka-Volterra systems (4.31) satisfy conditions above, the system (4.31) admits the first integral (4.53). □

We remark that the first integral given in Lemma 4.12 point 1 is the same integral that is given in the two-dimensional case. This can be guessed as the first two equations of (4.31) are independent of x_3.

We now turn to the case where there is at least one constant term equal to zero in the next three subsections. Here, we are going to use the integrating factor matrix T_2 as it no longer necessarily equal to zero in order to satisfy the conditions (4.44-4.46). The reason we do not use the integrating factor matrix T_1 is that we would have the same integrals as we have found before[3], except the fact that now there is at least one constant term equal to zero.

Recalling conditions (4.44-4.46) for the integrating factor matrix T_2, then all the param-

[3] since the l_i must all be equal to one due to conditions (4.48-4.50)

na.	definitions	na.	definitions
B_1	$b_1\alpha - b_3\gamma$	A_{1i}	$a_{1i}\alpha - a_{3i}\gamma$
B_2	$b_2\alpha + b_3\beta$	A_{2i}	$a_{2i}\alpha + a_{3i}\beta$
B_3	$b_1\beta + b_2\gamma$	A_{3i}	$a_{1i}\beta + a_{2i}\gamma$

Table 4.1: Definitions of the terminology used for the first integral conditions ($i = 1, 2, 3$).

eters must satisfy the following equations,

$$B_1 l_1 + B_2 l_2 = 0, \tag{4.80}$$
$$B_3 l_1 + B_2 l_3 = 0, \tag{4.81}$$
$$B_3 l_2 - B_1 l_3 = 0, \tag{4.82}$$
$$A_{13} l_1 + A_{23} l_2 = 0, \tag{4.83}$$
$$A_{32} l_1 + A_{22} l_3 = 0, \tag{4.84}$$
$$A_{31} l_2 - A_{11} l_3 = 0, \tag{4.85}$$
$$A_{11} l_1 + A_{21} l_2 = -A_{11}, \tag{4.86}$$
$$A_{12} l_1 + A_{22} l_2 = -A_{22}, \tag{4.87}$$
$$A_{31} l_1 + A_{21} l_3 = -A_{31}, \tag{4.88}$$
$$A_{33} l_1 + A_{23} l_3 = -A_{23}, \tag{4.89}$$
$$A_{32} l_2 - A_{12} l_3 = -A_{32}, \tag{4.90}$$
$$A_{33} l_2 - A_{13} l_3 = A_{13}, \tag{4.91}$$

and

$$\gamma e_2(l_2 - 1)/x_2 = \gamma e_3(l_3 - 1)/x_3 = (\beta l_2 - \alpha l_3)e_1/x_1 = 0, \tag{4.92}$$
$$\beta e_1(l_1 - 1)/x_1 = \beta e_3(l_3 - 1)/x_3 = (\alpha l_3 + \gamma l_1)e_2/x_2 = 0, \tag{4.93}$$
$$\alpha e_1(l_1 - 1)/x_1 = \alpha e_2(l_2 - 1)/x_2 = (\beta l_2 - \gamma l_1)e_3/x_3 = 0. \tag{4.94}$$

The notations A_{ij} and B_i above are defined in the table 4.1. Note that

$$B_1\beta + B_2\gamma - B_3\alpha = 0$$

as well as

$$A_{1i}\beta + A_{2i}\gamma - A_{3i}\alpha = 0, \quad \forall i.$$

4.3.2 The case $e_1, e_2 \neq 0, e_3 = 0$

Given that $e_1, e_2 \neq 0, e_3 = 0$, we conclude that $l_1 = l_2 = 1$ due to conditions (4.92-4.94), and l_3 must satisfy the following equations:

$$\beta - \alpha l_3 = \alpha l_3 + \gamma = 0. \tag{4.95}$$

The results are described in the following lemma.

Lemma 4.13. *If the vector field (4.31) with $e_1, e_2 \neq 0, e_3 = 0$ satisfies the conditions indicated in the table 4.2 then it has the following corresponding functions as a first integral respectively,*

1. $H = b_1x_1x_2 + a_{11}x_1^2x_2 - a_{22}x_1x_2^2 + e_1x_2 - e_2x_1$,
2. $H = (a_{13} - a_{23})x_1x_2x_3^2/2 + e_1x_2x_3 - e_2x_1x_3$,
3. $H = (e_1x_2 - e_2x_1)x_3^{l_3}$, *where l_3 is a solution of*

$$b_1 - b_3l_3 = a_{11} + a_{31}l_3 = a_{12} + a_{32}l_3 = a_{13} + a_{33}l_3 = 0.$$

case	conditions
1	$b_1 + b_2 = 2a_{11} + a_{21} = 2a_{22} + a_{12} = 0,\ a_{13} = a_{23} = 0$
2	$a_{13} - a_{23} \neq 0,\ b_1 + b_3 = b_2 + b_3 = 0,\ a_{21} + a_{31} = a_{11} - a_{21} = 0$
	$a_{22} + a_{32} = a_{12} - a_{32} = 0,\ a_{13} + a_{23} + 2a_{33} = 0$
3	$b_3^2 + a_{31}^2 + a_{32}^2 + a_{33}^2 \neq 0,\ b_1 - b_2 = 0,\ a_{1i} - a_{2i} = 0$
	$b_3a_{1i} - b_1a_{3i} = 0,\ i = 1, 2, 3$

Table 4.2: The first integral conditions for Lemma 4.13.

We also remark that the first integral given in Lemma 4.13 point 1 is trivial as it follows directly from the two-dimensional case.

4.3.3 The case $e_1 \neq 0, e_2 = e_3 = 0$

In this case, it immediately follows that $l_1 = 1$ due to conditions (4.93) and (4.94), and we have also the following equation to satisfy condition (4.92),

$$\beta l_2 - \alpha l_3 = 0. \tag{4.96}$$

All results for this case are given in the following lemma,

Lemma 4.14. *If the vector field (4.31) with $e_1 \neq 0, e_2 = e_3 = 0$ satisfies the conditions indicated in the table 4.3 then it has the following corresponding functions as a first integral (where A_{ij} and B_i are defined in table 4.1).*

1. $H = -B_2x_1 - A_{21}x_1^2/2 + \alpha e_1 \ln|x_2| + \beta e_1 \ln|x_3|$, *where α, β are solutions of $A_{22} = 0$, $A_{23} = 0$,*
2. $H = -a_{23}x_1x_3 + e_1 \ln|x_2|$,
3. $H = \beta \ln|x_3| + \alpha \ln|x_2|$, *where α, β are solutions of $B_2 = 0$ and $A_{2i} = 0$,*
4. $H = A_{33}x_1x_3 + \beta e_1 \ln|x_3| - \gamma e_1 \ln|x_2|$, *where β, γ are solutions of $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$,*
5. $H = A_{12}x_1x_2 + \gamma e_1 \ln|x_3| + \alpha e_1 \ln|x_2|$, *where α, γ are solutions of $B_1 = 0$, $A_{11} = 0$, $A_{13} = 0$,*

6. $H = (a_{13} + a_{23})x_3 - (a_{12} + a_{32})x_2 + e_1 \ln|x_3| - e_1 \ln|x_2|$,

7. $H = -a_{23}x_1x_2^{l_2}x_3 + e_1x_2^{l_2}/l_2$, *where* $l_2 = -(a_{13} + a_{33})/a_{23}$,

8. $H = (b_1 + a_{11}x_1)x_1x_2^{l_2}x_3^{l_3} + e_1x_2^{l_2}x_3^{l_3}$, *where* l_2, l_3 *are solutions of*

$$b_2l_2 + b_3l_3 = -b_1, a_{21}l_2 + a_{31}l_3 = -2a_{11}, a_{22}l_2 + a_{32}l_3 = a_{23}l_2 + a_{33}l_3 = 0$$

9. $H = A_{33}x_1x_2^{l_2}x_3^{l_3+1} + \beta(l_3 + 1)e_1x_2^{l_2}x_3^{l_3}/l_3$, *where* β, γ *are solutions of equations* $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$ *and* l_2, l_3 *are given below*

$$l_2 = -\frac{\alpha(a_{13} + a_{33})}{A_{33}} \quad l_3 = -\frac{\beta(a_{13} + a_{33})}{A_{33}},$$

10. $H = (a_{12} + a_{22})x_1x_2^{l_2+1}x_3^{l_3} + e_1x_2^{l_2}x_3^{l_3}$, *where* α, β *are solutions of equations* $B_2 = 0$, $A_{21} = 0$, $A_{23} = 0$ *and* l_2, l_3 *are given below*

$$l_2 = -\frac{\alpha(a_{12} + a_{22})}{A_{22}} \quad l_3 = -\frac{\beta(a_{12} + a_{22})}{A_{22}},$$

11. $H = ((a_{12}+a_{22})x_2/l_3+(a_{13}+a_{23})x_3/(l_3+1))x_1x_2^{l_2}x_3^{l_3}+e_1x_2^{l_2}x_3^{l_3}/l_3$, *where* $l_2 = -(a_{12}+ a_{22})/(a_{22} - a_{32})$ *and* $l_3 = -l_2$.

Note that in the table 4.3, we have used a square bracket notation to a matrix. As an example, in case 4 of table 4.3, $[B_3; A_{31}; A_{32}]$ is a 3×2 matrix formed from linear equations $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$, as follows

$$[B_3; A_{31}; A_{32}] = \begin{bmatrix} b_1 & b_2 \\ a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}. \tag{4.97}$$

We also note that the integrals in the lemma 4.14 point 2 and 6 do not depend on the variable x_1. They immediately follow from the two-dimensional case.

4.3.4 The case $e_1 = e_2 = e_3 = 0$

Finally, we shall discuss the case where all the constant terms are zero. The equations (4.92 - 4.94) are satisfied immediately. This means we only need to find conditions on the parameters in order to satisfy equations (4.80 - 4.91).

In the following lemma, we describe our results obtained using an integrating factor matrix of the form T_2.

Lemma 4.15. *If the vector field (4.31) with* $e_1 = e_2 = e_3 = 0$ *satisfies the conditions indicated in the table 4.4 then it has the following corresponding functions as a first integral (where* A_{ij} *and* B_i *are defined in table 4.1),*

1. $H = \alpha(b_1 \ln|x_2| - b_2 \ln|x_1| - a_{21}x_1) + \beta(b_1 \ln|x_3| - b_3 \ln|x_1| - a_{31}x_1)$ *where* α, β *are solutions of equations* $A_{22} = 0$ *and* $A_{23} = 0$,

case	conditions
1	$b_1 = a_{11} = a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{23}a_{32} = 0$
2	$b_2 = a_{21} = a_{22} = 0$, $a_{23} \neq 0$ and $b_1 + b_3 = a_{1i} + a_{3i} = 0$ $(i = 1, 2, 3)$
3	$\text{rank}[B_2; A_{21}; A_{22}; A_{23}] = 1$
4	$A_{33} \neq 0$, $b_1 + b_3 = a_{1i} + a_{3i} = 0$ $(i = 1, 2, 3)$
	and $\text{rank}[B_3; A_{31}; A_{32}] = 1$
5	$A_{12} \neq 0$, $b_1 + b_2 = a_{1i} + a_{2i} = 0$ $(i = 1, 2, 3)$
	and $\text{rank}[B_1; A_{11}; A_{13}] = 1$
6	$a_{12} + a_{32} \neq 0$, $a_{13} + a_{23} \neq 0$, $b_1 + b_2 = b_1 + b_3 = 0$,
	$a_{11} + a_{21} = a_{11} + a_{31} = 0$ and $a_{12} + a_{22} = a_{13} + a_{33} = 0$
7	$a_{23} \neq 0$, $a_{13} + a_{33} \neq 0$, $b_1 = a_{21} = a_{22} = 0$ and $b_1 + b_3 = a_{11} + a_{31} = a_{12} + a_{32} = 0$
8	$a_{12} = a_{13} = 0$, $a_{22}(-b_1a_{31} + 2b_3a_{11}) + a_{32}(-2b_2a_{11} + b_1a_{21}) = 0$
	and $a_{23}(-b_1a_{31} + 2b_3a_{11}) + a_{33}(-2b_2a_{11} + b_1a_{21}) = 0$
9	$A_{33} \neq 0$ $a_{13} + a_{33} \neq 0$, $b_1 + b_3 = a_{11} + a_{31} = a_{12} + a_{32} = 0$
	and $\text{rank}[B_3; A_{31}; A_{32}] = 1$
10	$A_{22} \neq 0$ $a_{12} + a_{22} \neq 0$, $b_1 + b_2 = a_{11} + a_{21} = a_{13} + a_{33} = 0$
	and $\text{rank}[B_2; A_{21}; A_{23}] = 1$
11	$a_{12} + a_{22} \neq 0$, $a_{22} - a_{32} \neq 0$, $a_{13} + a_{33} \neq 0$, $a_{23} - a_{33} \neq 0$,
	$b_1 + b_3 = b_2 - b_3 = 0$, $a_{11} + a_{31} = a_{21} - a_{31} = 0$
	and $(a_{13} + a_{33})(a_{22} - a_{32}) - (a_{12} + a_{22})(a_{23} - a_{33}) = 0$

Table 4.3: The first integral conditions for Lemma 4.14.

2. $H = \alpha(b_2/x_1 - a_{21}\ln|x_1| + a_{11}\ln|x_2|) + \beta(b_3/x_1 - a_{31}\ln|x_1| + a_{11}\ln|x_3|)$ *where* α, β *are solutions of equations* $A_{22} = 0$, $A_{23} = 0$,

3. $H = A_{31}\ln|x_3| + A_{11}\ln|x_2| - A_{21}\ln|x_1| + A_{22}x_2/x_1$ *where* α, β, γ *are solutions of* $B_1 = 0$, $A_{13} = 0$ *and* $\beta + \gamma = 0$,

4. $H = A_{31}\ln|x_3| + A_{11}\ln|x_2| - A_{21}\ln|x_1| + A_{33}x_3/x_1$ *where* α, β, γ *are solutions of* $B_2 = 0$, $A_{22} = 0$, *and* $\alpha - \gamma = 0$,

5. $H = -(a_{11} - a_{21})\ln|x_3| + (a_{11} - a_{31})\ln|x_2| - (a_{21} - a_{31})\ln|x_1| + (a_{22} - a_{32})x_2/x_1 - (a_{13} - a_{23})x_3/x_1$,

6. $H = x_1^{\beta}x_2^{\gamma}$, *where* β, γ *are solutions of* $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$ *and* $A_{33} = 0$,

7. $H = x_1^{\beta}x_2^{\gamma+\alpha}x_3^{\beta}$, *where* α, β, γ *are solutions of* $B_2 = 0$, $B_3 = 0$, $A_{2i} = 0$ *and* $A_{3i} = 0$ $(i = 1, 2, 3)$,

8a. $H = x_1^{l_1}x_2^{l_2}(A_{12}x_2/(l_2+1) + A_{33}x_3)$ *where* $l_1 = -\beta(-a_{23} + a_{33})/A_{33}$, $l_2 = -A_{13}/A_{33}$ *and* α, β, γ *are solutions of* $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$ *and* $\alpha + \beta = 0$,

8b. $H = x_1^{l_1}x_2^{l_2}(-b_3\beta/l_1 + A_{33}x_3)$ *where* $l_1 = -a_{33}\beta/A_{33}$, $l_2 = -(a_{33}\gamma)/A_{33}$ *and* β, γ *are solutions of* $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$, *8c.* $H = x_1^{l_1} x_2^{l_2}((a_{21} - a_{31})x_1/(l_1 + 1) + (a_{22} - a_{32})x_2/l_2 + (a_{13} - a_{23})x_3)$ *where* $l_1 = (a_{23} - a_{33})/(a_{13} - a_{23})$, $l_2 = (a_{33} - a_{13})/(a_{13} - a_{23})$,

8d. $H = x_1^{l_1} x_2^{l_2}(-A_{21}x_1/(l_1+1) + A_{33}x_3)$ *where* $l_1 = -A_{23}/A_{33}$, $l_2 = -\alpha(a_{33} - a_{13})/A_{33}$ *and* α, β, γ *are solutions of* $B_3 = 0$, $A_{31} = 0$, $A_{32} = 0$ *and* $\alpha - \gamma = 0$,

8e. $H = x_1^{l_1} x_2^{l_2}(-(b_2 + b_3)/(l_1) - (a_{21} + a_{31})x_1/l_2 + (a_{13} + a_{23})x_3)$, *where* $l_1 = -(a_{23} + a_{33})/(a_{13} + a_{23})$ *and* $l_2 = -(a_{33} - a_{13})(a_{13} + a_{23})$,

9a. $H = x_1^{l_1} x_2^{l_2} x_3^{l_3}(b_1 + a_{11}x_1)$ *where* $l_1 = -(a_{11}B_2)/(a_{11}B_2 - b_1 A_{21})$, $l_2 = -(b_1\alpha)/(B_2)$, $l_3 = (b_1\beta)/(B_2)$, *and* α, β *are solutions of* $A_{22} = 0$, $A_{23} = 0$,

9b. $H = x_1^{l_1} x_2^{l_2} x_3^{l_3}(a_{12}x_2/l_3 - a_{13}x_3/l_2)$ *where*

$$l_1 = \frac{(a_{22} - a_{32})(a_{23} - a_{33})}{-a_{12}(a_{23} - a_{33}) + a_{13}(a_{22} - a_{32})} \quad l_2 = \frac{a_{12}(a_{23} - a_{33})}{-a_{12}(a_{23} - a_{33}) + a_{13}(a_{22} - a_{32})},$$

and $l_2 + l_3 = -1$,

9c. $H = x_1^{l_1} x_2^{l_2} x_3^{l_3}((b_1 + b_2)/(l_3) + (a_{11} + a_{21})x_1/l_3 + (a_{12} + a_{22})x_2/l_3 - (a_{23} + a_{33})x_3/l_1)$ *where*

$$l_1 = \frac{A_{22}(A_{21} - A_{11})}{A_{11}A_{22} - A_{21}A_{12}} \quad l_2 = \frac{A_{11}(A_{12} - A_{22})}{A_{11}A_{22} - A_{21}A_{12}} \quad l_3 = \frac{A_{31}(A_{33} - A_{23})}{A_{31}A_{23} - A_{21}A_{33}},$$

where $A_{11}A_{22} - A_{21}A_{12} \neq 0$, $A_{31}A_{23} - A_{21}A_{33} \neq 0$ *and* $\alpha = \beta = \gamma$.

Remark that the integral in the Lemma 4.15 no 6. is the same integral we found in the two-dimensional LV case.

4.3.5 On the first integrals of three-dimensional Lotka-Volterra systems

We here give some remarks about lemma 4.15, in which we discussed first integrals of three-dimensional LV systems that have already been extensively discussed in the literature [14, 15, 23, 24, 37–39, 43, 46, 96, 102]. The form of first integrals obtained in lemma 4.15 (points 1-5) is similar to the ones obtained by Plank (1995) [102], even though the integrals he obtained are special cases [17] of invariants found in Cairó and Feix [15]. It is not hard to show that their integrals are different from the ones we have obtained in this chapter.

The remainder of the integrals in the lemma 4.15 (points 6-10) have the form

$$x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}(\varphi) \tag{4.98}$$

where φ is a polynomial function of degree one in x_1, x_2, x_3. To our best knowledge, this form first appeared as a first integral in Cairó and Feix [15], except the fact that the first integral in their paper is time-dependent. Through a time-rescaling (see Hua et al. [63]) they obtained a time-independent first integral. The existence of first integrals of this form of three-dimensional LV systems is also extensively investigated by Cairó [14]. He investigated a polynomial function φ of degree one and two. The integral functions in point 6-7 are similar to the ones he found [14, Theorem 2(1)]. The forms of integrals in point 8 do not seem to have been recognized before, thus these new results extend the known results on integrals of the form (4.98). The integral of the form 9a generalizes integrals obtained in [14, Theorem 2(8-13)]. Finally, integrals of the form 9b and 9c seem to be new.

case	conditions
1	$b_1 \neq 0$, $a_{11} = a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{32}a_{23} = 0$
2	$a_{11} \neq 0$, $b_1 = a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{32}a_{23} = 0$
3	$A_{11}^2 + A_{31}^2 \neq 0$, $A_{22} \neq 0$,
	$b_1 - b_2 = a_{12} - a_{22} = a_{13} - a_{23} = 0$ and $b_1a_{33} - b_3a_{13} = 0$
4	$A_{11}^2 + A_{31}^2 \neq 0$, $A_{33} \neq 0$,
	$b_1 - b_3 = a_{12} - a_{32} = a_{13} - a_{33} = 0$ and $b_2a_{32} - b_3a_{22} = 0$
5	$(a_{11} - a_{31})^2 + (a_{11} - a_{21})^2 \neq 0$, $a_{22} - a_{32} \neq 0$,
	$a_{23} - a_{33} \neq 0$ and $b_1 - b_3 = b_1 - b_2 = a_{12} - a_{22} = a_{13} - a_{33} = 0$
6	$\mathrm{rank}[B_3; A_{31}; A_{32}; A_{33}] = 1$
7	$\mathrm{rank}[B_3; A_{31}; A_{32}; A_{33}] = 1$ and $\mathrm{rank}[B_2; A_{21}; A_{22}; A_{23}] = 1$
8a	$A_{33} \neq 0$, $A_{13} \neq 0$, $A_{23} \neq 0$,
	$A_{12} \neq 0$, $b_2 - b_3 = a_{21} - a_{31} = 0$ and $\mathrm{rank}[B_3; A_{31}; A_{32}] = 1$
8b	$b_3 \neq 0$, $a_{33} \neq 0$, $A_{33} \neq 0$, $a_{31} = a_{32} = 0$,
	and $\mathrm{rank}[B_3; A_{31}; A_{32}] = 1$
8c	$a_{11} - a_{31} \neq 0$, $a_{12} - a_{32} \neq 0$, $a_{13} - a_{33} \neq 0$, $a_{23} - a_{33} \neq 0$, $a_{13} - a_{23} \neq 0$,
	$b_1 - b_2 = b_1 - b_3 = 0$ and $a_{11} - a_{21} = a_{12} - a_{22} = 0$
8d	$A_{33} \neq 0$, $A_{13} \neq 0$, $A_{23} \neq 0$,
	$A_{21} \neq 0$, $b_1 - b_3 = a_{12} - a_{32} = 0$ and $\mathrm{rank}[B_3; A_{31}; A_{32}] = 1$
8e	$a_{13} + a_{23} \neq 0$, $b_1 + b_2 = 0$, $a_{11} + a_{21} = 0$, $a_{12} + a_{22} = 0$, $a_{12} - a_{32} = 0$
	$(b_1 - b_3)a_{23} - (b_2 + b_3)a_{13} = 0$
9a	$b_1^2 + a_{11}^2 \neq 0$, $B_2 \neq 0$, $a_{11}B_2 - b_1A_{21} \neq 0$, $a_{12} = a_{13} = 0$
	and $a_{22}a_{33} - a_{32}a_{23} = 0$
9b	$a_{13} \neq 0$, $a_{23} \neq 0$, $a_{22} - a_{32} \neq 0$, $a_{23} - a_{33} \neq 0$, $b_1 = a_{11} = 0$
	and $b_2 - b_3 = a_{21} - a_{31} = 0$
9c	$B_1A_{22}(A_{21} - A_{11}) + B_2A_{11}(A_{12} - A_{22}) = 0$, $A_{13}A_{22}(A_{21} - A_{11})$
	$+A_{23}A_{11}(A_{12} - A_{22}) = 0$, $B_3A_{23}(A_{21} - A_{31}) + B_2A_{31}(A_{33} - A_{23}) = 0$
	$A_{32}A_{23}(A_{21} - A_{31}) + A_{22}A_{31}(A_{33} - A_{23}) = 0$,
	with $\alpha = \beta = \gamma$

Table 4.4: The first integral conditions for Lemma 4.15.

4.4 Discussion

In this chapter, we have derived first integrals for two and three-dimensional Lotka-Volterra systems with constant terms through an integrating factor matrix. We make an Ansatz and conditions for the existence of first integrals are obtained. Moreover, the question of the existence of an integral in dynamical systems has been changed into a linear algebra problem.

In the two-dimensional case, the integrating factor matrix $T(x_1, x_2)$ in equation (4.4), along with the condition such that T is an integrating factor, turns out to be similar to the one that is used by Plank [102]. This is because in the two-dimensional case we have $S = T^{-1}$. In the three-dimensional case this property no longer applies, as a 3×3 skew-symmetric matrix is not invertible. Hence, our analysis is more general than Plank's, as we did not assume that there is an isolated equilibrium, as was assumed in his paper.

Comparison with Darboux method The Darboux method has been applied to find first integrals of two-dimensional [20, 22] and three-dimensional [14, 23] LV systems. Compared to this method, our method has advantages in the context of searching for a first integral of Lotka-Volterra systems with constant terms. To apply the Darboux method, one must seek an algebraic curve of a vector field, and Lotka-Volterra systems without constant terms have a natural algebraic curve as the systems have invariant axis. For systems with constant terms, this no longer applies, as the presence of the constant terms make the invariant axis vanish.

Comparison with Hamiltonian method The existence of first integrals of two-dimensional systems has been obtained using Hamiltonian method [16, 19, 63]. They assumed that the integrals are products of two or three polynomial functions of degree one. Thus, our method has advantages in terms of the number of Ansatzs, as we only assume one form of Ansatz. Gao [37] has also applied a Hamiltonian method to find first and second integrals of special cases of three-dimensional LV systems, where the linear terms are absent.

Comparison with Frobenius method The Frobenius method was first introduced by Strelcyn and Wojciechowski [115] to find a first integral for three-dimensional systems. It has been used to find integrals for LV systems by Grammaticos et al. [46]. Unfortunately, they only look at a special case of the LV system, which is the so-called ABC system.

Comparison with Carleman embedding method The existence of first integrals in n dimensions has been studied [15, 18] through the Carleman embedding method. However, the integrals that are obtained are time-dependent, which is sometimes called an *invariant*.

For future work, it would be a challenging problem to find a more general integrating factor matrix of a vector field in dimension greater than or equal to two. People have obtained first integrals and conditions for the existence of such integrals and we could use their results to endeavour to find the general integrating factor matrix.

CHAPTER 5

Conclusion

In this final chapter, a summary of all results presented in this thesis is discussed. The discussion focuses on different aspects of all our analyses of the Lotka-Volterra system with constant terms. First we focus on the bifurcation analysis that we have done in the chapters two and three of this thesis. We shall discuss the non-versal unfolding aspect of unusual bifurcations that we have found in the two-dimensional Lotka-Volterra system with a constant term in chapter two. We also discuss the idea of bifurcation analysis of systems with a special structure. The discussion then turns to the issue of the existence of first integrals of two- and three-dimensional Lotka-Volterra systems with constant terms.

5.1 Bifurcation analysis of Lotka-Volterra systems with a constant term

In chapter two, we have performed bifurcation analysis of the two-dimensional Lotka-Volterra system with a constant term. In the model (2.1), there are seven parameters available but we have three continuous symmetries which are used to reduce the number of parameters to be varied from seven to four. However, we only vary two when performing bifurcation analysis, as the other two are used to distinguish the topological difference of bifurcation diagrams. The bifurcation diagrams are very useful in classifying the dynamics of the system we analyse in terms of parameters that we vary. The bifurcation curves in the bifurcation diagram are boundaries of different dynamics that can occur. We then can interpret the dynamics in each area of each bifurcation diagram in terms of the original problem.

We have performed not only bifurcation analysis but also investigated how degenerate our bifurcations are. In particular, we are interested in bifurcations where saddle-node and transcritical bifurcations interact. It turns out that these unusual bifurcations can be considered as non-versal unfoldings of common bifurcations in general systems which are a codimension-two cusp bifurcation and a codimension-three degenerate Bogdanov-Takens bifurcation. The cusp bifurcation has been known for a long time. Since the normal form of a cusp bifurcation is one-dimensional, the degeneracy of cusp bifurcation naturally appears when varying parameters. The cusp bifurcation is one of seven elementary catastrophes listed by Thom [118] along with the simple saddle-node bifurcation. In contrast, a codimension-three degenerate Bogdanov-Takens bifurcation is still a new theory and rarely occurs in application.

The study of the degenerate Bogdanov-Takens bifurcation is interesting on its own. Kuznetsov [79] has derived a practical computation to find the coefficient of the normal form of the Bogdanov-Takens bifurcation. Therefore, we can detect if a Bogdanov-Takens bifurcation is degenerate and how degenerate it is. Although the global analysis of such a bifurcation is extensively investigated in Dumortier et al. [31], and all references therein, the occurrence of this bifurcation is rarely found in applications. To our best knowledge, Baer et al. [6] is one of the first to find such a bifurcation in an application, i.e. in a model for a two-stage structured population.

Our approach to understand unusual bifurcations that we have found is to prove that these bifurcations are actually non-versal unfoldings of common bifurcations such as cusp and Bogdanov-Takens bifurcations. We have introduced minimal models that undergo the same bifurcations as we have. We also have tried to find non-linear smooth maps that respectively transform our model to the minimal model and transform the minimal model to the normal form of a generic bifurcation that is already known. However, in this chapter we did not answer the question of why, in the first place, our model has such unusual bifurcations. This question has been answered in the next chapter where we discuss systems with a special structure.

5.2 Dynamical systems with a special structure

In chapter three we have discussed dynamical systems with a codimension-one invariant manifold. We have applied bifurcation analysis to those systems, imposing that the analysis preserves the property of the special structure. We started with the simplest degeneracy that a system can have which is the single-zero eigenvalue. It turns out that we have a transcritical bifurcation instead of a more generic saddle-node bifurcation. Another consequence of the codimension-one invariant manifold in the codimension-one degeneracy is the fact that we cannot have a Hopf bifurcation.

Continuing to higher codimension bifurcations (i.e. codimension-two bifurcations), we have analysed more degenerate singularities that are similar to cusp, Bogdanov-Takens and fold-Hopf bifurcations respectively. The degeneracies that we have discussed are similar to those generic codimension-two bifurcations, but give us different sets of bifurcations as a result of the special structure that our system has, which is a codimension-one invariant manifold. In the first case, the cusp bifurcation is unfolded in a different way such that we have found an interaction of saddle-node and transcritical bifurcations. However, in the double-zero degeneracy, the Bogdanov-Takens bifurcation is not unfolded differently, instead we come to a more degenerate codimension-three Bogdanov-Takens bifurcation. This can be checked by computing the normal form coefficients of this bifurcation.

We now want to discuss the connection between the bifurcation analysis of dynamical systems with a special structure and the two-dimensional Lotka-Volterra system with a constant term. The bifurcation structure of the Lotka-Volterra system contains two important degenerate bifurcations that we have studied in this thesis in chapter two, namely the single-zero eigenvalue degeneracy combined with second order degeneracy and the double-zero eigenvalue degeneracy. Both of these degeneracies are shown in chapter three to produce the same set of bifurcations as the Lotka-Volterra system does. Therefore, the Lotka-Volterra system with a constant term has such unusual bifurcations mainly because it has a codimension-one invariant manifold structure.

Thus, we have two approaches in explaining bifurcations in the Lotka-Volterra system. One is to understand generic bifurcations and to try to relate them to bifurcations in our system. The other approach is to understand the special structure that our system has, in this case the special structure is the invariant manifold. Between these two approaches, the latter seems more common as the analysis of a special structure has its own place in dynamical system theory. For example, the study of dynamical system with a special structure which is a symmetry has been studied in great detail [82].

Another special structure that may be pursued in future work is a codimension-two invariant manifold in our system. The manifold that is discussed in this thesis is one dimension less than the space we are working with, thus there are some bifurcations that are just not allowed to occur such as the Hopf bifurcation as a codimension-one bifurcation, a degenerate Hopf, and a double-Hopf bifurcations as codimension-two bifurcations. If we can reduce the dimension of the invariant manifold, we shall have more exciting bifurcations as more bifurcations are allowed.

5.3 The existence of first integrals of Lotka-Volterra systems with constant terms

The existence of first integrals of Lotka-Volterra systems with constant terms has been discussed in chapter four. Various authors have obtained first integrals and conditions on the parameters for the Lotka-Volterra system without the constant terms, while we have found first integrals and conditions on the parameters for the Lotka-Volterra system including additional constant terms. The integrating factor matrix method has been used to reduce the problem of searching for a first integral to a linear algebra problem. We want to mention that the conditions for the Lotka-Volterra system with constant terms do not depend on the constant terms. As a consequence, the model has a first integral for all values of the constant terms.

The question that we then want to discuss is what the first integral function and its constraint are used for. As mentioned in the beginning of the chapter four, finding first integrals is a classical tool in the classification of phase portraits of a dynamical system. Not only that but finding first integrals gives global information about the long-term behaviour of dynamical systems. Another reason why the existence of such a function is studied, is to find the location of global bifurcations in the parameter space. This analysis is actually discussed in chapter three. The Melnikov method that is applied to a differential equation, needs a reduced integrable model of the same differential equation as a starting point for a perturbation method in finding the location of heteroclinic or homoclinic bifurcations. It is true that recently, the detection of global bifurcation has used a numerical continuation package like AUTO2000 (see Doedel et al. [29]). However, it is always nice to compute the formula for global bifurcations analytically.

Another reason why first integrals of differential equations are important is because the first integral is used in the theory of numerical methods to solve differential equations, in particular in geometrical integration. We refer to Quispel and McLachlan [104] for a more detailed explanation of the theory of geometrical integration. Geometric integration is the term used to describe numerical methods for computing the solution of differential equations, while preserving one or more physical/mathematical properties of the system exactly (i.e. up to roundoff error). If a dynamical system has a first integral, then the integral can be used in *test equations* for geometrical numerical integration methods. For instance, we refer to Nakamura and Hashimoto [99] in which a comparison is made between geometrical numerical integration schemes applied to three-dimensional Lotka-Volterra systems.

In the discussion of chapter four, we have compared our methods with known methods to find first integrals such as the Carleman embedding method, the Frobenius method, the Darboux method and the Hamiltonian method. Each of these methods obtained a first integral along with its conditions on the parameters. Using their results, it would be a challenging problem to find a more general integrating factor matrix of a vector field in dimension greater than or equal to two.

We also remark that searching for a first integral does not always give a completely integrable case. It is true that in two-dimensional systems, the existence of a first integral implies that the system is completely integrable because the phase portraits are completely characterized. However, in three-dimensional systems (or greater than three) this is not the case. Thus, there is still a problem to find a second integral. Note that the existence of a second integral gives a completely integrable case in three-dimensional vector fields.

In the end, to prove the integrability or to prove the existence of integrals of general differential equations is very hard and no systematic methods to solve it are known. All attempts described above do not completely solve this integrability problem but represent approaches in this direction.

5.4 Final words

After almost a century, the Lotka-Volterra model is still intriguing many mathematicians. This model has been used to approximate many applications in the real world and thus there have been a large number of modifications to this model such that the physical problem can be approximated as closely as possible. In addition to real world applications, this model is also interesting from a mathematical point of view, as many aspects and theories in mathematics have been applied to understand the behaviour of the solution of this model. The conditions for this system to be integrable and the bifurcation analysis of this system that have been introduced in this thesis are just some of a number of mathematical investigations on this model around the world.

Dynamical systems theory, which has provided tools to analyse such systems in this thesis, is interesting in its own right. This theory has seen an explosive growth and expansion in the past 50 years. Integrability and bifurcation analysis are among methods of global analysis in dynamical systems theory. Finally, we cannot restrict ourselves to dynamical systems theory to analyse such system, as the Lotka-Volterra system has also attracted much attention in other branches of mathematics.

APPENDIX A

Proof of Proposition (3.6)

We recall the proposition (3.6),

The center manifold of a single-zero eigenvalue or a pair of purely imaginary eigenvalue of the matrix (3.4) lies inside the codimension-one invariant manifold up to any desired degree of accuracy.

We shall prove this proposition. First we shall consider the case when the matrix (3.4) has a single zero eigenvalue. We already assumed that the origin is an equilibrium. We recall the equation (3.2),

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_{n-1}, y), \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_{n-1}, y), \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}, y), \\
\dot{y} &= y f_n(x_1, x_2, \ldots, x_{n-1}, y).
\end{aligned}$$

Without loss of generality, we assume that the Jacobian matrix of the equation above evaluated at the origin has been transformed into a (real) Jordan canonical form. Suppose that the single-zero eigenvalue occurs in the x_1-direction, then by the center manifold theorem applied to the coordinates $(x_1, \ldots, x_{n-1})$, we can assume that the vector field is two-dimensional. Thus, we have

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, y), \\
\dot{y} &= y f_2(x_1, y),
\end{aligned} \tag{A.1}$$

where f_1 is nonlinear in x_1 and y. By using the center manifold theorem, there is a function $y = h(x_1)$ such that the dynamics restricted to the center manifold is given by,

$$\dot{x}_1 = f_1(x_1, h(x_1)).$$

We want to prove that the center manifold lies entirely inside the codimension-one invariant manifold $y = 0$, thus we need to show $h(x_1) = 0$. We assume $h(x_1)$ has the form,

$$h(x_1) = h_2 x_1^2 + h_3 x_1^3 + h_4 x_1^4 + \ldots .$$

We Taylor expand the functions $f_1(x_1, y)$ and $f_2(x_1, y)$,

$$\begin{aligned}
f_1(x_1, y) &= f_1^{(20)} x_1^2 + f_1^{(11)} x_1 y + f_1^{(02)} y^2 + \ldots, \\
f_2(x_1, y) &= f_2^{(00)} + f_2^{(10)} x_1 + f_2^{(01)} y + \ldots,
\end{aligned}$$

where $f_2^{(00)} \neq 0$ since the Jacobian matrix of the system (A.1) has only a single-zero eigenvalue. We finally compute the derivative of $y = h(x_1)$ with respect to time,

$$\begin{aligned} \dot{y} &= \tfrac{dh}{dx_1}\dot{x}_1, \\ yf_2(x_1, y) &= (2h_2x_1 + 3h_3x_1^2 + 4h_4x_1^3 + \ldots)f_1(x_1, y), \\ h(x_1)f_2(x_1, h(x_1)) &= (2h_2x_1 + 3h_3x_1^2 + 4h_4x_1^3 + \ldots)f_1(x_1, h(x_1)). \end{aligned}$$

Using the Taylor expansion of the functions $f_1(x_1, y)$ and $f_2(x_1, y)$, we can compute the coefficients of the function $h(x_1)$ by equating coefficients of x_1 that have the same power. The smallest power of x_1 in the left hand side is two, while that of the right hand side is three. This gives $h_2 = 0$. The fact that $h_2 = 0$ implies that the smallest power of x_1 in the left hand side is three, while that of the right hand side is now four. This now gives $h_3 = 0$. It will continue this way, in fact, we will get $h_i = 0$ for $i = 2, 3, \ldots$. This implies that the center manifold is $y = 0$, which is the same equation as the invariant manifold. Thus, the first part of the proposition is proven.

We now want to prove the second part which is to show that the center manifold coming from a pair of purely imaginary eigenvalue degeneracy also lies inside the invariant manifold up to any desired degree of accuracy. However, we shall only prove that the center manifold lies inside the invariant manifold up the third order term.

Without loss of generality, we assume that the Jacobian matrix evaluated at the origin has been transformed into a (real) Jordan canonical form. By using the center manifold theorem applied to the coordinate $(x_1, \ldots, x_{n-1})$, we can assume that the vector field is three-dimensional. Thus, we have

$$\begin{aligned} \dot{x}_1 &= -x_2 + f_1(x_1, x_2, y), \\ \dot{x}_2 &= x_1 + f_2(x_1, x_2, y), \\ \dot{y} &= yf_3(x_1, x_2, y), \end{aligned}$$

where f_1,f_2 are nonlinear in x_1,x_2 and y and $f_3(0, 0, 0)$ is not zero. By the center manifold theorem, there is a function $y = h(x_1, x_2)$. We want to prove that $h(x_1, x_2) = 0$. First we assume that $h(x_1, x_2)$ has the following form,

$$h(x_1) = h_{20}x_1^2 + h_{11}x_1x_2 + h_{02}x_2^2 + \ldots .$$

Computing the derivative of $y = h(x_1, x_2)$ with respect to time gives,

$$\begin{aligned} \dot{y} &= \tfrac{\partial h}{\partial x_1}\dot{x}_1 + \tfrac{\partial h}{\partial x_2}\dot{x}_2, \\ yf_3(x_1, x_2, y) &= \tfrac{\partial h}{\partial x_1}(-x_2 + f_1(x_1, x_2, y)) + \tfrac{\partial h}{\partial x_2}(x_1 + f_2(x_1, x_2, y)), \\ h(x_1, x_2)f_3(x_1, x_2, h(x_1, x_2)) &= \tfrac{\partial h}{\partial x_1}(-x_2 + f_1(x_1, x_2, h(x_1, x_2))) + \tfrac{\partial h}{\partial x_2}(x_1 + f_2(x_1, x_2, h(x_1, x_2))). \end{aligned}$$

By Taylor expanding the functions $f_1(x_1, x_2, y)$, $f_2(x_1, x_2, y)$, and $f_3(x_1, x_2, y)$, we are able to compute the coefficient of the function $h(x_1, x_2)$ by equating coefficients that have the same power in terms of x_1 and x_2. Equating the second order terms gives,

$$\begin{aligned} (x_1^2): \quad h_{20}f_3(0, 0, 0) &= h_{11}, \\ (x_1x_2): \quad h_{11}f_3(0, 0, 0) &= -2h_{20} + 2h_{02}, \\ (x_2^2): \quad h_{02}f_3(0, 0, 0) &= -h_{11}. \end{aligned}$$

The only solution of the linear system above is $(h_{20}, h_{11}, h_{02}) = (0, 0, 0)$, which is what we want. We then compute the coefficients of third order,

$$\begin{aligned}
(x_1^3): \quad h_{30} f_3(0,0,0) &= h_{21}, \\
(x_1^2 x_2): \quad h_{21} f_3(0,0,0) &= -3h_{30} + 2h_{12}, \\
(x_1 x_2^2): \quad h_{12} f_3(0,0,0) &= -2h_{21} + 3h_{03}, \\
(x_2^3): \quad h_{03} f_3(0,0,0) &= -h_{12},
\end{aligned}$$

which gives trivial solutions $(h_{30}, h_{21}, h_{12}, h_{02}) = (0, 0, 0, 0)$. Thus, we have proven that the center manifold coming from a pair of purely imaginary eigenvalue degeneracy lies inside the invariant manifold up to the third order term.

APPENDIX B

Bifurcation analysis of the Lotka-Volterra system with a constant term

In this appendix, we shall check conditions of bifurcations that occur in the Lotka-Volterra system with a constant term (2.1). As depicted in Figure 2.1, numerous bifurcations occur in this system. The conditions for saddle-node and transcritical bifurcations that are obtained analytically are going to be discussed in this appendix.

B.1 Bifurcation conditions of the first saddle-node bifurcation

We first check conditions for the first saddle-node bifurcation to occur. The saddle-node bifurcation occurs when the coordinates and the parameters are:

$$\begin{aligned} &x_1^* = -\frac{b_1}{2a_{11}}, \quad e^* = \frac{b_1^2}{4a_{11}^2}, \\ &x_2^* = 0, \end{aligned} \tag{B.1}$$

where $a_{11} \neq 0$. Along the line $e = b_1^2/(4a_{11}^2)$, the Jacobian matrix of the system (2.1) evaluated at this equilibrium has two eigenvalues, which are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = b_2 - \frac{b_1 a_{21}}{2a_{11}}.$$

We assume that $\lambda_2 = b_2 - b_1 a_{21}/(2a_{11}) \neq 0$, since the only degeneracy is a single-zero eigenvalue. We translate the critical equilibrium to the origin and consider e as one of the coordinates by introducing a new coordinate system, $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$ and $y_3 = e - e^*$, thus we have

$$\begin{aligned} \dot{y}_1 &= \gamma y_2 + y_3 + a_{11} y_1^2 + a_{12} y_1 y_2, \\ \dot{y}_2 &= (b_2 - \frac{b_1 a_{21}}{2a_{11}}) y_2 + a_{21} y_1 y_2 + a_{22} y_2^2, \\ \dot{y}_3 &= 0, \end{aligned} \tag{B.2}$$

where $\gamma = -b_1 a_{12}/(2a_{11})$. To check the saddle-node bifurcation conditions we need to first diagonalize the system above and to apply the center manifold theorem to reduce the diagonalized system restricted to the center manifold. A transformation given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 2a_{11}\gamma & 0 \\ 0 & 2b_2a_{11} - b_1a_{21} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}, \tag{B.3}$$

will bring the system (B.2) to the following diagonalized system,

$$\begin{aligned} \dot{u}_1 &= \alpha + a_{11}u_1^2 + C_1u_1u_2 + C_2u_2^2, \\ \dot{u}_2 &= (b_2 - \frac{b_1a_{21}}{2a_{11}})u_2 + a_{21}u_1u_2 + C_3u_2^2, \\ \dot{\alpha} &= 0, \end{aligned} \tag{B.4}$$

where

$$\begin{aligned} C_1 &= -2a_{11}a_{21}\gamma + a_{21}(2b_2a_{11} - b_1a_{21})4a_{11}^2\gamma, \\ C_2 &= 2a_{11}\gamma(2a_{11}^2\gamma - 2a_{21}a_{11}\gamma - a_{22}(2b_2a_{11} - b_1a_{21}) + a_{12}(2b_2a_{11} - b_1a_{21})), \\ C_3 &= (2a_{11}a_{21}\gamma + (2b_2a_{11} - b_1a_{21})a_{22}). \end{aligned}$$

By using the center manifold theorem, there is a function $u_2 = h(u_1, \alpha) \in \mathcal{O}(\|(u_1, \alpha)\|^2)$ for a sufficiently small neighbourhood $(u_1, \alpha) = (0, 0)$ such that the dynamics restricted to the center manifold is represented by,

$$\dot{u}_1 = f(\alpha, u_1) = \alpha + a_{11}u_1^2 + C_1u_1h(u_1, \alpha) + C_2h(u_1, \alpha)^2. \tag{B.5}$$

The degeneracy condition for saddle-node bifurcation is,

$$\frac{\partial f}{\partial u_1}(0, 0) = 0, \tag{B.6}$$

while the non-degeneracy conditions are

$$\frac{\partial f}{\partial \alpha}(0, 0) = 1, \quad \text{and} \quad \frac{\partial^2 f}{\partial u_1^2} = 2a_{11}. \tag{B.7}$$

We conclude that the Lotka-Volterra system with a constant term undergoes a codimension-one saddle-node bifurcation along the line $e = b_1^2/(4a_{11})$ with $a_{11} \neq 0$. However, there are some values of the parameters such that the bifurcation is degenerate which is when $(b_2 - b_1a_{21}/(2a_{11})) = 0$, in which we have a double-zero eigenvalues degeneracy.

B.2 Bifurcation conditions of the second saddle-node bifurcation

The second saddle-node bifurcation occurs when

$$\begin{aligned} x_1^* &= \rho, & e^* &= \frac{(-b_1a_{22} + b_2a_{12})^2}{4a_{22}D_1}, \\ x_2^* &= \frac{-b_2 - a_{21}\rho}{a_{22}}, \end{aligned} \tag{B.8}$$

where $D_1 = a_{11}a_{22} - a_{12}a_{21}$ and $\rho = (-b_1a_{22} + b_2a_{12})/(2D_1)$. We assume that $D_1, a_{22} \neq 0$. Along the line $e = (-b_1a_{22} + b_2a_{12})^2/(4a_{22}D_1)$, the Jacobian matrix of the system (2.1) evaluated at this equilibrium has two eigenvalues, which are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \frac{a_{12}a_{21}\rho - b_2a_{22} - a_{21}a_{22}\rho}{a_{22}},$$

where we assume λ_2 is not zero as there is no other degeneracy. After an initial transformation, given by $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$ and $y_3 = e - e^*$, the Lotka-Volterra system with a constant term can be written as the extended system,

$$\begin{aligned}
\dot{y}_1 &= \frac{a_{12}a_{21}}{a_{22}}y_1 + a_{12}\rho y_2 + y_3 + a_{11}y_1^2 + a_{12}y_1y_2, \\
\dot{y}_2 &= -\frac{a_{22}}{a_{21}}(b_2 + a_{21}\rho)y_1 - (b_2 + a_{21}\rho)y_2 + a_{21}y_1y_2 + a_{22}y_2^2, \\
\dot{y}_3 &= 0.
\end{aligned} \tag{B.9}$$

We diagonalize the system above using the following transformation,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12}\rho & 0 \\ -a_{21} & (b_2 + a_{21}\rho) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}, \tag{B.10}$$

thus we now have

$$\begin{aligned}
\dot{u}_1 &= \alpha - \frac{D_1(b_2 + a_{21}\rho)}{\lambda_2}u_1^2 - \frac{a_{11}a_{22}}{\lambda_2(b_2 + a_{21}\rho)}\alpha^2 + p_1u_1\alpha + p_2u_2\alpha + p_3u_1u_2 + p_4u_2^2, \\
\dot{u}_2 &= \lambda_2u_2 - \frac{D_1a_{21}}{\lambda_2}u_1^2 - \frac{a_{11}a_{22}a_{21}}{\lambda_2(b_2 + a_{21}\rho)^2}\alpha^2 + q_1u_1\alpha + q_2u_2\alpha + q_3u_1u_2 + q_4u_2^2, \\
\dot{\alpha} &= 0,
\end{aligned} \tag{B.11}$$

where

$$\begin{aligned}
p_1 &= -\frac{D_2}{\lambda_2} + \frac{a_{12}a_{21}^2\rho}{\lambda_2(b_2 + a_{21}\rho)}, & q_1 &= -\frac{a_{21}D_3}{\lambda_2(b_2 + a_{21}\rho)}, \\
p_2 &= -\frac{a_{12}}{\lambda_2}(-(2a_{11} - a_{21})\rho + (b_2 + a_{21}\rho)), & q_2 &= \frac{2a_{11}a_{12}a_{21}\rho}{\lambda_2(b_2 + a_{21}\rho)} - \frac{a_{21}(a_{12} + a_{22})}{\lambda_2}, \\
p_3 &= b_2a_{12} + \frac{2a_{12}\rho D_1(b_2 + a_{21}\rho)}{a_{22}\lambda_2}, & q_3 &= \frac{a_{21}(b_2 + a_{21}\rho)(a_{22} - a_{12})}{\lambda_2} + \frac{a_{12}a_{21}\rho D_3}{a_{21}\lambda_2}, \\
p_4 &= -\frac{a_{12}\rho(b_2 + a_{21}\rho)}{a_{22}\lambda_2}(a_{12}\rho(a_{11} - a_{21}) & q_4 &= \frac{1}{a_{22}\lambda_2}(-a_{11}a_{21}a_{12}^2\rho^2 - a_{22}^2(b_2 + a_{21}\rho)^2 \\
&\quad + (b_2 + a_{21}\rho)(a_{22} - a_{12})), & &\quad + a_{12}a_{21}\rho(b_2 + a_{21}\rho)(a_{12} + a_{22})),
\end{aligned}$$

with $D_2 = 2a_{11}a_{22} - a_{12}a_{21}$ and $D_3 = 2a_{11}a_{22} - a_{22}a_{21} - a_{12}a_{21}$. This system has a two-dimensional center manifold which can be represented locally as the graph of a polynomial function $u_2 = h(u_1, \alpha)$. The function $h(u_1, \alpha)$ is assumed to have the following form,

$$h(u_1, \alpha) = h_{20}u_1^2 + h_{11}u_1\alpha + h_{02}\alpha^2 + \mathcal{O}(\|(u_1, \alpha)\|^2). \tag{B.12}$$

We are not going to compute the coefficient of the polynomial above because we only want to prove the saddle-node bifurcation conditions. Thus, the dynamics restricted to the center manifold is represented by the following system,

$$\begin{aligned}
\dot{u}_1 &= f(u_1, \alpha) \\
&= \alpha - \frac{D_1(b_2 + a_{21}\rho)}{\lambda_2}u_1^2 - \frac{a_{11}a_{22}}{\lambda_2(b_2 + a_{21}\rho)}\alpha^2 + p_1u_1\alpha + p_2\alpha h(u_1, \alpha) \\
&+ \; p_3u_1h(u_1, \alpha) + p_4h(u_1, \alpha)^2.
\end{aligned} \tag{B.13}$$

The degeneracy condition of this bifurcation is

$$\frac{\partial f}{\partial u_1}(0,0) = 0, \tag{B.14}$$

while the non-degeneracy conditions are

$$\frac{\partial f}{\partial \alpha}(0,0) = 1, \quad \text{and} \quad \frac{\partial^2 f}{\partial u_1^2} = -2\frac{D_1(b_2 + a_{21}\rho)}{\lambda_2}. \tag{B.15}$$

We now conclude that the Lotka-Volterra system with a constant term undergoes another codimension-one saddle-node bifurcation along the line $e = (-b_1a_{22} + b_2a_{12})^2/(4a_{22}D_1)$ with $a_{22}D_1 \neq 0$. This bifurcation is degenerate when the term $(b_2 + a_{21}\rho)$ is zero, in which we have a degenerate saddle-node bifurcation with a degeneracy in the second order term or $\lambda_2 = (a_{12}a_{21}\rho - b_2a_{22} - a_{21}a_{22}\rho)/a_{22}$ is zero in which we have a double-zero eigenvalues degeneracy.

B.3 Bifurcation conditions of the transcritical bifurcation

The transcritical bifurcation occurs when

$$\begin{aligned} x_1^* &= \frac{b_2}{a_{21}}, \quad e^* = \frac{b_2(-b_2a_{11} + b_1a_{21})}{{a_{21}}^2}, \\ x_2^* &= 0, \end{aligned} \tag{B.16}$$

We assume that $a_{21} \neq 0$. Along the line $e = (b_2(-b_2a_{11} + b_1a_{21}))/a_{21}^2$, the Jacobian matrix of the system (2.1) evaluated at this equilibrium has two eigenvalues, which are

$$\lambda_1 = b_1 - \frac{2b_2a_{11}}{a_{21}} \quad \text{and} \quad \lambda_2 = 0.$$

We assume that $\lambda_1 = b_1 - (2b_2a_{11})/a_{21} \neq 0$ as there is no other degeneracy. We note that this assumption is equivalent with the assumption that we have in the first saddle-node bifurcation condition in the appendix B.1. We first transform the critical equilibrium to the origin by the transformation $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$ and $y_3 = e - e^*$. We thus have the new extended vector field as follows

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 - (\frac{b_2a_{12}}{a_{21}})y_2 + y_3 + a_{11}y_1^2 + a_{12}y_1y_2 \\ \dot{y}_2 &= a_{21}y_1y_2 + a_{22}y_2^2 \\ \dot{y}_3 &= 0. \end{aligned} \tag{B.17}$$

We diagonalize the above vector field by the following linear transformation,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & b_2a_{12} & a_{21} \\ 0 & (b_1a_{21} - 2b_2a_{11}) & 0 \\ 0 & 0 & -(b_1a_{21} - 2b_2a_{11}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}, \tag{B.18}$$

to get the following vector field,

$$\begin{aligned} \dot{u}_1 &= \lambda_1 u_1 + a_{11}u_1^2 + (b_1 - b_2)a_{12}a_{21}u_1u_2 + (b_1 - b_2)a_{12}a_{21}^2u_2u_3 + 2a_{11}a_{21}u_1u_3 \\ &\quad + a_{11}a_{21}^2u_3^2 + b_2a_{12}(b_2D_1 + (a_{22} - a_{12})(b_2a_{11} - b_1a_{21}))u_2^2 \\ \dot{u}_2 &= a_{21}^3\alpha u_2 + a_{21}u_1u_2 - 2D_1(b_2 + a_{21}\rho)u_2^2 \\ \dot{\alpha} &= 0, \end{aligned} \tag{B.19}$$

where $D_1 = a_{11}a_{22} - a_{12}a_{21}$ and $\rho = (-b_1a_{22} + b_2a_{12})/(2D_1)$ as in the previous section. The system above has a two-dimensional center manifold which can be represented locally as the graph of a polynomial function $u_1 = h(u_2, \alpha)$. The function $h(u_2, \alpha)$ is assumed to have the form

$$h(u_2, \alpha) = h_{20}u_2^2 + h_{11}u_2\alpha + h_{02}\alpha^2 + \mathcal{O}(\|(u_2, \alpha)^3\|). \tag{B.20}$$

Thus, the dynamics restricted to the center manifold is represented by the following equation,

$$\dot{u}_2 = a_{21}^2\alpha u_2 - 2D_1(b_2 + a_{21}\rho)u_2^2 + a_{21}u_2h(u_2, \alpha). \tag{B.21}$$

We set the right hand side of the one-dimensional vector field above as $f(u_2, \alpha)$ to check the transcritical bifurcation degeneracy conditions

$$\frac{\partial f}{\partial u_2}(0,0) = 0, \quad \frac{\partial f}{\partial \alpha}(0,0) = 0, \tag{B.22}$$

and the non-degeneracy conditions respectively,

$$\frac{\partial^2 f}{\partial u_2 \partial \alpha}(0,0) = a_{21}^2, \quad \frac{\partial^2 f}{\partial u_2^2}(0,0) = -4D_1(b_2 + a_{21}\rho), \tag{B.23}$$

since the function $h(u_2, \alpha)$ is at least of second order in terms of u_2 and α. Therefore we conclude that a codimension-one transcritical bifurcation occurs along the line $e = (b_2(-b_2a_{11} + b_1a_{21}))/a_{21}^2$ with non-zero a_{21}. This bifurcation is degenerate when either of these two following conditions occur:

1. $\lambda_1 = b_1 - (2b_2a_{11})/a_{21}$ is zero, or

2. $(b_2 + a_{21}\rho)$ is zero.

We note that if the first condition holds that we have a degenerate codimension-two bifurcation at which the transcritical bifurcation condition coincides with the first saddle-node bifurcation condition to form a double-zero degeneracy. While, if the second condition holds the transcritical bifurcation condition coincides with the second saddle-node bifurcation condition to form a codimension-two bifurcation with a single-zero eigenvalue combined with a degeneracy in the second order term.

APPENDIX C

Degeneracy and non-degeneracy conditions for bifurcations in Chapter 3

C.1 Bifurcations of the normal form with a double-zero eigenvalues degeneracy

In this section, we will check bifurcation conditions of the saddle-node and the transcritical bifurcation that occur in the unfolding of the normal form of the double-zero eigenvalues degeneracy in the system (3.40) in section 3.4.

C.1.1 Bifurcation conditions of the saddle-node bifurcation

First we shall check bifurcation conditions for the codimension-one saddle-node bifurcation that occurs when

$$(x, y) = (0, 0) \quad \text{and} \quad \mu_1 = 0. \tag{C.1}$$

Along the line $\mu_1 = 0$, the Jacobian matrix of the system (3.40) has two eigenvalues, which are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \mu_2. \tag{C.2}$$

We extend the vector field (3.40) to be three-dimensional by considering the parameter μ_1 as one of the coordinates, thus we have

$$\begin{aligned} \dot{x} &= \mu_1 + y + ax^2, \\ \dot{y} &= y(\mu_2 + bx), \\ \dot{\mu}_1 &= 0. \end{aligned} \tag{C.3}$$

We diagonalize the system above by the following transformation

$$\begin{pmatrix} x \\ y \\ \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}, \tag{C.4}$$

to get the following new system

$$\begin{aligned}\dot{u}_1 &= \alpha + au_1^2 + (2a-b)u_1u_2 + (a-b)u_2^2,\\ \dot{u}_2 &= \mu_2 u_2 + bu_1u_2 + bu_2^2,\\ \dot{\alpha} &= 0.\end{aligned} \tag{C.5}$$

By applying the center manifold theorem, there is a function $u_2 = h(u_1, \alpha)$ that is of at least second order such that the dynamics restricted to the center manifold is represented by

$$\dot{u}_1 = f(u_1, \alpha) = \alpha + au_1^2 + (2a-b)u_1u_2 + (a-b)u_2^2. \tag{C.6}$$

We do not compute the coefficient of the function $u_2 = h(u_1, \alpha)$ as the saddle-node bifurcation condition can be determined without computing this function. The degeneracy condition of the saddle-node bifurcation is

$$\frac{\partial f}{\partial u_1}(0,0) = 0, \tag{C.7}$$

while the non-degeneracy conditions are

$$\frac{\partial f}{\partial \alpha}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial u_1^2}(0,0) = 2a. \tag{C.8}$$

Therefore we conclude that a codimension-one saddle-node bifurcation occurs along the line $\mu_1 = 0$. This bifurcation is degenerate when $\mu_2 = 0$ in which we shall have a double-zero eigenvalue degeneracy.

C.1.2 Bifurcation conditions of the transcritical bifurcation

We shall check the system (3.40) that undergoes a transcritical bifurcation when

$$(x, y) = (-\frac{\mu_2}{b}, 0) \quad \text{and} \quad \mu_1 + \frac{a}{b^2}\mu_2^2 = 0. \tag{C.9}$$

Along the curve $\mu_1 + a/b^2\mu_2^2 = 0$, the Jacobian matrix of system (3.40) evaluated at this fixed point has two eigenvalues, which are

$$\lambda_1 = -\frac{2a\mu_2}{b} \quad \text{and} \quad \lambda_2 = 0.$$

We now translate the critical equilibrium above, also we consider the parameter μ_1 as one of the coordinates,

$$z_1 = x + \frac{\mu_2}{b}, \quad z_2 = y \quad \text{and} \quad z_3 = \mu_1 + \frac{a}{b^2}\mu_2^2.$$

Thus, we have the new extended vector field as follows,

$$\begin{aligned}\dot{z}_1 &= -\frac{2a\mu_2}{b}z_1 + z_2 + z_3 + az_1^2,\\ \dot{z}_2 &= bz_1z_2,\\ \dot{z}_3 &= 0.\end{aligned} \tag{C.10}$$

Again, we diagonalize the system above by the following linear transformation,

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ 0 & 2a\mu_2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}. \tag{C.11}$$

Thus we have,

$$\begin{aligned}
\dot{u}_1 &= -\frac{2a\mu_2}{b}u_1 + au_1^2 - \frac{b^2}{2a\mu_2}u_1\alpha - \frac{b^3}{2a\mu_2}u_2\alpha + b(2a-b)u_1u_2 + b^2(a-b)u_2^2, \\
\dot{u}_2 &= \frac{b^2}{2a\mu_2}u_2\alpha + b^2u_2^2 + \frac{b}{2a\mu_2}u_1\alpha + bu_1u_2, \\
\dot{\alpha} &= 0.
\end{aligned} \tag{C.12}$$

The system above has a two-dimensional center manifold that is represented by a graph of a polynomial function $u_1 = h(u_2, \alpha)$ that is of at least second order in terms of u_1 and α. Then the dynamics restricted to the center manifold can be represented by the following one-dimensional vector field,

$$\dot{u}_2 = -\frac{b^2}{2a\mu_2}u_2\alpha + b^2u_2^2 + \frac{b}{2a\mu_2}\alpha h(u_2,\alpha) + bu_2h(u_2,\alpha). \tag{C.13}$$

Setting the right hand side of the system above as $f(u_2, \alpha)$, we compute the degeneracy conditions of this bifurcation

$$\frac{\partial f}{\partial u_2}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(0,0) = 0,$$

while the non-degeneracy conditions are given by,

$$\frac{\partial^2 f}{\partial u_2^2}(0,0) = 2b^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial u_2 \partial \alpha}(0,0) = \frac{b^2}{2a\mu_2}.$$

Therefore, we conclude that a codimension-one transcritical bifurcation occurs in the unfolding of the normal form with double-zero eigenvalue degeneracy. This bifurcation is degenerate when $\mu_2 = 0$ at which we have a double-zero eigenvalue degeneracy.

C.2 Bifurcations of the normal form with a single-zero and a pair of purely imaginary eigenvalues degeneracies

In this section, we check the degeneracy and non-degeneracy conditions of bifurcations in the unfolding of normal forms with a single-zero and a pair of purely imaginary eigenvalues (3.84) in section 3.5. There are four bifurcations that we shall check, namely two transcritical bifurcations and two pitchfork bifurcations.

C.2.1 The first pitchfork bifurcation

The pitchfork bifurcation occurs when

$$(r, y) = (0, 0) \quad \text{and} \quad \mu_1 = 0$$

Along the line $\mu_1 = 0$, the Jacobian matrix of the system (3.84) has two eigenvalues:

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \mu_2.$$

We extend the system (3.84) by including the parameter μ_1 as one of the coordinates,

$$\begin{aligned}
\dot{r} &= \mu_1 r + a_1 ry + a_2 r^3, \\
\dot{y} &= y(\mu_2 - y - r^2). \\
\dot{\mu}_1 &= 0,
\end{aligned} \tag{C.14}$$

with non-zero μ_2, a_1 and a_2. The system above is already in the diagonal form, thus there is a two-dimensional center manifold represented by the graph of a polynomial function $y = h(r, \mu_1)$. The function $y = h(r, \mu_1)$ is of at least second order in terms of the coordinate y and the parameter μ_1. The dynamics of the system above restricted to the center manifold is given by:

$$\dot{r} = f(r, \mu_1) = \mu_1 r + a_1 r h(r, \mu_1) + a_2 r^3. \tag{C.15}$$

We want to compute the degeneracy and non-degeneracy conditions of the pitchfork bifurcation, which include computations of the third derivative of f with respect to r. For this reason we shall compute the second order coefficients of the center manifold function $h(r, \mu_1)$. First we assume that $h(r, \mu_1)$ has the following form,

$$y = h(r, \mu_1) = h_{20} r^2 + h_{11} r \mu_1 + h_{02} \mu_1^2 + \dots. \tag{C.16}$$

We compute the derivative of both sides of the equation above with respect to time,

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial r}(r, \mu_1)\dot{r} + \frac{\partial h}{\partial \mu_1}(r, \mu_1)\dot{\mu}_1, \qquad (C.17) \\ \mu_2 y - y^2 - r^2 y &= (2h_{20} r + h_{11}\mu_1 + \dots)(\mu_1 r + a_1 r y + a_2 r^3) + 0, \\ \mu_2 h(r, \mu_1) - h^2(r, \mu_1) - r^2 h(r, \mu_1) &= (2h_{20} r + h_{11}\mu_1 + \dots)(\mu_1 r + a_1 r h(r, \mu_1) + a_2 r^3). \end{aligned}$$

We equate the coefficients of $r^i \mu_1^j$ with $i + j \geq 2$. However, the smallest power in the right hand side of the equation above is three, while the smallest power in the left hand side of the equation above is two. Thus we can conclude that $h_{20} = h_{11} = h_{02} = 0$. This implies that the vector field restricted to the center manifold is given by,

$$\dot{r} = f(r, \mu_1) = \mu_1 r + a_2 r^3 + \mathcal{O}(\|(r, \mu_1)\|^4).$$

The degeneracy conditions of the system above are,

$$\frac{\partial f}{\partial r}(0,0) = 0, \quad \frac{\partial f}{\partial \mu_1}(0,0) = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial r^2}(0,0) = 0,$$

and the non-degeneracy conditions are

$$\frac{\partial^2 f}{\partial r \partial \mu_1}(0,0) = 1 \quad \text{and} \quad \frac{\partial^3 f}{\partial r^3}(0,0) = 6a_2.$$

Therefore we conclude that a codimension-one pitchfork bifurcation occurs in the system (3.84). We require that $\mu_2 \neq 0$ such that this bifurcation is non-degenerate.

C.2.2 The second pitchfork bifurcation

The second pitchfork bifurcation occurs in the system (3.84) when

$$(r, y) = (0, \mu_2) \quad \text{and} \quad \mu_1 + a_1 \mu_2 = 0.$$

Along the line $\mu_1 + a_1 \mu_2 = 0$, the Jacobian matrix of the system (3.84) has two eigenvalues:

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -\mu_2.$$

We bring the critical fixed point to the origin and consider μ_1 as one of the coordinates, by using the transformation $z_1 = r$, $z_2 = y - \mu_2$ and $z_3 = \mu_1 + a_1\mu_2$. Thus we have

$$\begin{aligned} \dot{z}_1 &= z_1 z_3 + a_1 z_1 z_2 + a_2 z_1^3, \\ \dot{z}_2 &= -\mu_2 z_2 - \mu_2 z_1^2 - z_2^2 - z_1^2 z_2, \\ \dot{z}_3 &= 0. \end{aligned} \tag{C.18}$$

The system above is already in the diagonal form, then there is a two-dimensional center manifold represented by a graph of a polynomial function $z_2 = h(z_1, \mu_1)$ that is of at least second order in terms of the coordinate z_1 and the parameter z_3. Thus the dynamics of the system above restricted to the center manifold is represented by the following vector field,

$$\dot{z}_1 = f(z_1, z_3) = z_1 z_3 + a_2 z_1^3 + a_1 z_1 h(z_1, \mu_1). \tag{C.19}$$

As in the previous section, one of the non-degeneracy conditions of the pitchfork bifurcation involves a third derivative of f with respect to z_1. Thus we have to compute the coefficients of the terms $z_1^i z_3^j$ with $i + j = 2$. We assume that the function $z_2 = h(z_1, z_3)$ has the following form,

$$h(z_1, z_3) = h_{20} z_1^2 + h_{11} z_1 z_3 + h_{02} z_3^2 + \dots,$$

and we compute the derivative of $z_2 = h(z_1, z_3)$ with respect to time,

$$\begin{aligned} \dot{z}_2 &= \frac{\partial h}{\partial z_1}(z_1 z_3)\dot{z}_1 + \frac{\partial h}{\partial z_3}(z_1 z_3)\dot{z}_3, \\ -\mu_2 z_2 - \mu_2 z_1^2 - z_2^2 - z_1^2 z_2 &= (2h_{20} z_1 + h_{11} z_3 + \dots)(z_1 z_3 + a_1 z_1 z_2 \\ &\quad + a_2 z_1^3) + 0, \\ -\mu_2 h(z_1, z_3) - \mu_2 z_1^2 - h^2(z_1, z_3) - z_1^2 h(z_1, z_3) &= (2h_{20} z_1 + h_{11} z_3 + \dots)(z_1 z_3 \\ &\quad + a_1 z_1 h(z_1, z_3) + a_2 z_1^3). \end{aligned} \tag{C.20}$$

Equating the coefficient of $z_1^i z_3^j$ with $i + j = 2$ that has the same power on both sides we can compute the coefficients of the polynomial function $h(z_1, z_2)$,

$$\begin{aligned} (z_1^2): \quad -\mu_2 h_{20} - \mu_2 &= 0, \\ -\mu_2 h_{20} &= \mu_2, \\ h_{20} &= -1, \\ (z_1 z_3): \quad -\mu_2 h_{11} &= 0, \\ h_{11} &= 0, \\ (z_3^2): \quad -\mu_2 h_{02} &= 0, \\ h_{02} &= 0. \end{aligned} \tag{C.21}$$

Thus, the system (C.19) now reads,

$$\dot{z}_1 = z_3 z_1 + (a_2 - a_1) z_1^3 + \mathcal{O}(\|(z_1, z_3)\|^4). \tag{C.22}$$

Therefore, the degeneracy conditions of the pitchfork bifurcation are,

$$\frac{\partial f}{\partial z_1}(0,0) = 0, \quad \frac{\partial f}{\partial z_3}(0,0) = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial z_1^2}(0,0) = 0,$$

while, the non-degeneracy conditions are

$$\frac{\partial^2 f}{\partial z_1 \partial z_3}(0,0) = 1 \quad \text{and} \quad \frac{\partial^3 f}{\partial z_1^3}(0,0) = 6(a_2 - a_1).$$

The term $(a_2 - a_1)$ is not zero as it has been assumed in the text in section 3.5. We conclude that a codimension-one pitchfork bifurcation occurs in the system (3.84). This bifurcation is degenerate when both μ_1 and μ_2 are zero.

C.2.3 The first transcritical bifurcation

The system (3.84) also undergoes a transcritical bifurcation. The fixed point and the value of the parameters are

$$(r, y) = (0,0) \quad \text{and} \quad \mu_2 = 0.$$

Along the line $\mu_2 = 0$, the Jacobian matrix of the system (3.84) has two eigenvalues:

$$\lambda_1 = \mu_1 \quad \text{and} \quad \lambda_2 = 0.$$

We include the parameter μ_2 as one of of the coordinates, thus we have:

$$\begin{aligned} \dot{r} &= \mu_1 r + a_1 r y + a_2 r^3, \\ \dot{y} &= y(\mu_2 - y - r^2), \\ \dot{\mu}_2 &= 0. \end{aligned} \tag{C.23}$$

The system above is already in the diagonal form. Thus, there is a two-dimensional center manifold that is represented by a graph of a polynomial function $r = h(y, \mu_2)$. This function is at least second order in terms of the coordinate y and the parameter μ_2. Hence the dynamics of the system above restricted to the center manifold is given by,

$$\dot{y} = \mu_2 y - y^2 + y h^2(y, \mu_2). \tag{C.24}$$

Thus, setting the right hand side of the equation above as $f(y, \mu_2)$,the degeneracy conditions for the transcritical bifurcation of the system (3.84) are

$$\frac{\partial f}{\partial y}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \mu_2}(0,0) = 0,$$

while the non-degeneracy conditions of the transcritical bifurcation are

$$\frac{\partial^2 f}{\partial y^2}(0,0) = -2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial \mu_2}(0,0) = 1.$$

Thus we conclude that a codimension-one transcritical bifurcation occurs in the system (3.84) when $\mu_2 = 0$. The only point where this bifurcation is degenerate is $(\mu_1, \mu_2) = (0,0)$.

C.2.4 The second transcritical bifurcation

The system (3.84) also undergoes a second transcritical bifurcation when the coordinates of the fixed point and the values of the parameters are

$$(r, y) = (\sqrt{\mu_2}, 0) \quad \text{and} \quad \mu_1 + a_2 \mu_2 = 0,$$

where μ_2 is assumed to be positive. Along the line $\mu_1 + a_2\mu_2 = 0$, the Jacobian matrix of the system (3.84) has two eigenvalues, which are:

$$\lambda_1 = 2a\mu_2 \quad \text{and} \quad \lambda_2 = 0.$$

We bring the critical fixed point to the origin and consider the parameter μ_1 as one of the coordinates using the following transformation, $z_1 = r - \sqrt{\mu_2}$ $z_2 = y$ and $z_3 = \mu_1 + a_2\mu_2$. Thus we have

$$\begin{aligned}
\dot{z}_1 &= 2a_2\mu_2 z_1 + a_1\sqrt{\mu_2}z_2 + \sqrt{\mu_2}z_3 + a_1 z_1 z_2 + z_1 z_3 + 3a_2\sqrt{\mu_2}z_1^2 + a_2 z_1^3, \\
\dot{z}_2 &= -z_2^2 - 2\sqrt{\mu_2}z_1 z_2 - z_1^2 z_2, \\
\dot{z}_3 &= 0.
\end{aligned} \tag{C.25}$$

We diagonalize the system above using the following transformation,

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & a_1 & 1 \\ 0 & -2a_2\sqrt{\mu_2} & 0 \\ 0 & 0 & -2a_2\sqrt{\mu_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}. \tag{C.26}$$

Thus, we now have,

$$\begin{aligned}
\dot{u}_1 &= 2a_2\mu_2 u_1 + \varphi(u_1, u_2, \alpha), \\
\dot{u}_2 &= -2\sqrt{\mu_2}\alpha u_2 + 2\sqrt{\mu_2}(a_2 - a_1)u_2^2 - \alpha^2 u_2 - a_1^2 u_2^3 - 2\alpha u_1 u_2 \\
&\quad - 2\sqrt{\mu_2}u_1 u_2 - 2a_1 u_1 u_2^2 - 2a_1\alpha u_2^2 - u_1^2 u_2, \\
\dot{\alpha} &= 0.
\end{aligned} \tag{C.27}$$

where the function φ is given by,

$$\begin{aligned}
\varphi(u_1, u_2, \alpha) &= a_2\sqrt{\mu_2}\alpha^2 + 3a_2\sqrt{\mu_2}u_1^2 + a_1\sqrt{\mu_2}(a_1 a_2 - 2(a_2 - a_1))u_2^2 + 4a_2\sqrt{\mu_2}\alpha u_1 \\
&\quad + 2\sqrt{\mu_2}a_1(a_2 + 1)\alpha u_2 + 2\sqrt{\mu_2}a_1(2a_2 + 1)u_1 u_2 + a_2 u_1^3 + a_1(3a_2 + 1)u_1^2 u_2 \\
&\quad + 3a_2\alpha u_1^2 + a_1^2(3a_2 + 2)u_1 u_2^2 + 2a_1(3a_2 + 1)\alpha u_1 u_2 + 3a_2\alpha^2 u_1 + a_1^3(a_2 + 1)u_2^3 \\
&\quad + a_1^2(3a_2 + 2)\alpha u_2^2 + a_2\alpha^3 + a_1(3a_2 + 1)\alpha^2 u_2.
\end{aligned} \tag{C.28}$$

The dynamics of the vector field above has a two-dimensional center manifold that is represented by a graph of a polynomial function $u_1 = h(u_2, \alpha)$. Then the dynamics restricted to the center manifold is given by the following vector field,

$$\begin{aligned}
\dot{u}_2 &= f(u_2, \alpha), \\
&= -2\sqrt{\mu_2}\alpha u_2 + 2\sqrt{\mu_2}(a_2 - a_1)u_2^2 - \alpha^2 u_2 - a_1^2 u_2^3 - 2\alpha u_2 h(u_2, \alpha) \\
&\quad - 2\sqrt{\mu_2}u_2 h(u_2, \alpha) - 2a_1 u_2^2 h(u_2, \alpha) - 2a_1\alpha u_2^2 - u_2 h^2(u_2, \alpha).
\end{aligned} \tag{C.29}$$

The degeneracy conditions for the transcritical bifurcation are

$$\frac{\partial f}{\partial u_2}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(0,0) = 0,$$

while the non-degeneracy conditions are

$$\frac{\partial^2 f}{\partial u_2 \partial \alpha}(0,0) = -2\sqrt{\mu_2} \quad \text{and} \quad \frac{\partial^2 f}{\partial u_2^2}(0,0) = 2\sqrt{\mu_2}(a_2 - a_1).$$

The term $(a_2 - a_1)$ is not zero as it has been assumed in section 3.5. Then we conclude that a codimension-one transcritical bifurcation occurs in the system (3.84) along the line $\mu_1 + a_2\mu_2 = 0$. The only point where this bifurcation is degenerate occurs when $(\mu_1, \mu_2) = (0, 0)$.

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